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EFFECT OF VARIABLE VISCOSITY AND THERMAL CONDUCTIVITY ON HIGH-SPEED SLIP FLOW BETWEEN CONCENTRIC CYLINDERS [1]

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SUMMARY

Schamberg was the first to solve the differential equations of slip flow, including the Burnett terms, for concentric circular cylinders assuming constant coefficients of viscosity and thermal conductivity. The problem is solved for variable coefficients of viscosity and thermal conductivity in this paper by applying a transformation which leads to an iteration method. Starting with the solution for constant coefficients, this method enables one to approximate the solution for variable coefficients very closely after one or two steps. Satisfactory results are shown to follow from Schamberg's solution by using his values of the constant coefficients multiplied by a constant factor η, leading to what are denoted as the effective coefficients of viscosity and thermal conductivity.

INTRODUCTION

The fact that a gas is not a continuum but actually a collection of molecules in rapid but random motion has begun to have more and more importance in the aerodynamics of high-speed flow. This is due to the expectation that flow through wind tunnels at low pressure or flight at extremely high altitudes will not be amenable to analysis using classical fluid dynamics. When the mean free path of the molecules l is negligible compared with the macroscopic dimension L, which may be wing chord, tunnel diameter, and so forth, the classical picture should hold as the molecules are so tightly packed together the gas behaves just like a mathematical continuum. The ratio l/L is defined as the Knudsen number Kn, which is a measure of the degree of gas rarefaction. In terms of the better known parameters Reynolds number Re and Mach number M, the Knudsen number is proportional to M/Re. Hence, although not a new parameter, it is a convenient one to use when the degree of rarefaction of the gas is of interest.

Gas dynamics is the continuous-flow regime or Clausius gas regime for which the Navier-Stokes equations together with the condition of no slip on the boundaries are valid and the Knudsen number is extremely small. If the gas becomes more rarefied and the Knudsen number increases, the effect of slip along the boundaries becomes noticeable, although the Navier-Stokes equations remain valid so long as the Mach number remains small. This phenomenon has been known for over 75 years and has been the subject of an extensive study by physicists. Tsien (ref. 1) has summarized this work very well. During this same period of time the solution of Boltzmann's integral equation by Enskog and Chapman, along lines laid down by Hilbert, has led to the distribution function for a nonuniform gas as an expansion in powers of the Knudsen number. This approach yields the equations of flow in successive orders of approximation, the first order being the Navier-Stokes equations, the second order, the Burnett equations, and so forth. The third-order approximation has never been carried out and the expected complexity of the result does not seem to make the attempt worth while, especially as the restrictions on the properties of the gas itself are not strictly valid. Chapman and Cowling (ref. 2) have presented this theory in their well-known treatise.

Tsien (ref. 1) presented the Burnett equations of motion and pointed out that unless the product Mach number times Knudsen number ($M\ Kn$) was significant any problem in flow could be theoretically solved using the Navier-Stokes equations. The question of the proper boundary conditions when the higher order Burnett terms are included was raised by Tsien but not answered until 2 years later when Schamberg, one of Tsien's students, showed in his doctor's thesis (ref. 3) that the number of boundary conditions required for the Burnett equations is the same as for the Navier-Stokes equations.[2] While being the same in number, the Schamberg boundary conditions are considerably more complex, being also expansions in powers of the Knudsen number. The first approximations for the slip velocity and temperature jump remain essentially the same as those used by the physicists in their treatment of low-speed slip flows (ref. 4, ch. 8). The second approximation which is required when used in conjunction with the Burnett equations for high-speed flows is new and, like the Burnett terms, of considerable complexity.

The Burnett equations and the Schamberg boundary conditions apply to the domain of high-speed slip flow that the aerodynamicist expects to enter first when he leaves the domain of classical gas dynamics. Their great complexity discourages expectation of a theoretical solution of any practically important problem. Hence, the solution of any problem, even trivial so far as flows go, is difficult; but, if the problem can be set up experimentally, the attempt would seem worth while in order to determine the validity of the expansions in powers of Knudsen number and a possible delineation of the dividing line between gas dynamics and slip flow.

[1] Supersedes NACA TN 2895, "Effect of Variable Viscosity and Thermal Conductivity on High-Speed Slip Flow Between Concentric Cylinders" by T. C. Lin and R. E. Street, 1953.

[2] The correctness of Schamberg's boundary conditions is not universally accepted, but they are the only ones proposed so far.

Experimental results have now been obtained for flow past spheres (ref. 5) and transverse flow past circular cylinders (ref. 6) both of which are almost insurmountable theoretical problems. Schamberg (ref. 3) solved the plane Couette flow problem and the problem of the simple rotation between two coaxial cylinders. In order to linearize his equations he assumed that the coefficient of viscosity and the coefficient of heat conduction of the gas were absolute constants. Lin (ref. 7) removed the restriction of constant values of these coefficients and recalculated the plane Couette flow for a perfect gas with constant specific heats and constant Prandtl number but with the coefficients of viscosity and heat conduction varying as a constant power of the absolute temperature. The present investigation does the same for the flow between coaxial cylinders. In contrast with the plane Couette flow problem the flow between two coaxial cylinders rotating relative to each other seems of more than academic interest, since an experimental check is quite possible and no doubt will be performed in the near future.

The problem is set up in its general form assuming only that the Burnett equations and Schamberg boundary conditions are valid and the flow is steady and stable. Thus the streamlines are circles and only the flow in a single plane normal to the cylindric axis need be considered. Whether such a flow can be stable at high rotary speeds is outside the domain of the method used here. A consideration of the stability criteria based upon the Navier-Stokes equations with slip at the boundary is to be found in reference 8. While special laws of dependence of the gas properties are assumed, the method is theoretically possible for other laws as well as for variable specific heats and Prandtl number.

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FUNDAMENTAL EQUATIONS AND EXPRESSIONS FOR STRESS TENSOR AND HEAT-FLUX VECTOR

It is convenient to start from the general equations of the mean motion of a fluid all of whose physical properties vary; in Cartesian tensor notation these equations are (ref. 2, 7, or 9):

The continuity equation:

$$\frac{D\rho}{Dt}+\rho\frac{\partial u_i}{\partial x_i}=0 \tag{1}$$

The momentum equation:

$$\rho\frac{Du_i}{Dt}-\rho F_i+\frac{\partial P_{ij}}{\partial x_j}=0 \tag{2}$$

The energy equation:

$$\rho\frac{DE}{Dt}+P_{ij}\frac{\partial u_i}{\partial x_j}+\frac{\partial q_i}{\partial x_i}=0 \tag{3}$$

where ρ and E are, respectively, the density and the internal energy per unit mass; x_i is the Cartesian coordinate in the physical space; u_i, F_i, and q_i are, respectively, the components of the velocity of the fluid mean motion, the external force per unit mass, and the heat-flux vector in the x_i-direction; P_{ij} is the component of the pressure tensor; and

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+u_i\frac{\partial}{\partial x_i} \tag{4}$$

is the comoving time derivative or time derivative following the motion as in hydrodynamics. (See appendix A for definitions of all symbols.) The summation convention, summing over repeated subscripts, is used.

These general equations can be derived directly from Maxwell's equation of transfer by making use of the properties of the summational invariants for molecular encounters without determining the form of the molecular-velocity-distribution function. The more convenient Cartesian tensor notation is used rather than the vector-dyadic notation preferred by Chapman and Cowling (ref. 2, pp. 51 and 52).

In terms of the stress tensor τ_{ij}, which is defined by

$$P_{ij}=p\delta_{ij}+\tau_{ij} \tag{5}$$

(p being the hydrostatic pressure and δ_{ij}, the unit tensor), the momentum equation, equation (2), and the energy equation, equation (3), take the following forms, respectively:

$$\rho\frac{Du_i}{Dt}-\rho F_i+\frac{\partial p}{\partial x_i}+\frac{\partial \tau_{ij}}{\partial x_j}=0 \tag{6}$$

$$\rho\frac{DE}{Dt}+p\frac{\partial u_i}{\partial x_i}+\tau_{ij}\frac{\partial u_i}{\partial x_j}+\frac{\partial q_i}{\partial x_i}=0 \tag{7}$$

Upon using the continuity equation, equation (1), and the first law of thermodynamics

$$dQ=dE+p\,d\left(\frac{1}{\rho}\right) \tag{8}$$

together with the definitions $dS=dQ/T$ and $H=E+(p/\rho)$, Q, S, and H being the heat received, the entropy, and the enthalpy per unit mass of the gas, respectively, it is easy to show that

$$\rho\frac{DE}{Dt}+p\frac{\partial u_i}{\partial x_i}=\rho T\frac{DS}{Dt}=\rho\frac{DH}{Dt}-\frac{Dp}{Dt} \tag{9}$$

and energy equation (7) becomes

$$\rho T \frac{DS}{Dt}+\tau_{ij}\frac{\partial u_i}{\partial x_j}+\frac{\partial q_i}{\partial x_i}=0 \tag{10}$$

or

$$\rho \frac{DH}{Dt}-\frac{Dp}{Dt}+\tau_{ij}\frac{\partial u_i}{\partial x_i}+\frac{\partial q_i}{\partial x_i}=0 \tag{11}$$

Adding to equation (11) the product of u_i and equation (6) and considering that τ_{ij} is a symmetrical tensor, another form of the energy equation is obtained:

$$\rho \frac{D}{Dt}\left(H+\frac{1}{2}u_iu_i\right)-\rho u_iF_i-\frac{\partial p}{\partial t}+\frac{\partial q_i}{\partial x_i}+\frac{\partial}{\partial x_i}(\tau_{ij}u_j)=0 \tag{12}$$

Equations (3), (7), (10), and (11) are the most familiar forms of the energy equation. Neglecting terms containing the external force F_i and setting H equal to c_pT, equations (6) and (12) yield the general momentum and energy equations given by Tsien (ref. 1) and used by Schamberg (ref. 3). The sign of τ_{ij}, being the same as that used by Tsien and Schamberg, is opposite to the usual convention.

It is only through the expressions for the stress tensor τ_{ij} and the heat-flux vector q_i that the above momentum and energy equations depend on the form of the molecular-velocity-distribution function. Let ${}_r\tau_{ij}$ and ${}_rq_i$ denote the rth-order approximations to the stress tensor τ_{ij} and the heat-flux vector q_i, respectively, and write

$${}_r\tau_{ij}=\sum_{n=0}^{r}\tau_{ij}^{(n)} \tag{13}$$

and

$${}_rq_i=\sum_{n=0}^{r}q_i^{(n)} \tag{14}$$

where $\tau_{ij}^{(n)}$ and $q_i^{(n)}$ are the nth-order corrections to ${}_0\tau_{ij}$ and ${}_0q_i$, respectively. Then the first-order approximation to the molecular-velocity-distribution function, that is, the Maxwellian distribution, gives (ref. 2, pp. 112, 122, and 123)

$${}_0\tau_{ij}=\tau_{ij}^{(0)}=0 \tag{15}$$

and

$${}_0q_i=q_i^{(0)}=0 \tag{16}$$

which, together with equations (6) and (7), yield the Eulerian equation of motion

$$\rho \frac{Du_i}{Dt}-\rho F_i+\frac{\partial p}{\partial x_i}=0 \tag{17}$$

and the corresponding energy equation

$$\rho \frac{DE}{Dt}+p\frac{\partial u_i}{\partial x_i}=0 \tag{18}$$

The second-order approximation to the molecular-velocity-distribution function gives (ref. 2, pp. 112, 122, and 123)

$${}_1\tau_{ij}=\tau_{ij}^{(1)}=-2\mu\overline{\frac{\partial u_i}{\partial x_j}} \tag{19}$$

and

$${}_1q_i=q_i^{(1)}=-\lambda\frac{\partial T}{\partial x_i} \tag{20}$$

where μ and λ are the coefficients of viscosity and thermal conductivity, respectively, and $\overline{\frac{\partial u_i}{\partial x_j}}$ is the nondivergent symmetrical tensor associated with the tensor $\frac{\partial u_i}{\partial x_j}$, that is,

$$\overline{\frac{\partial u_i}{\partial x_j}}\equiv\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)-\frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}$$

In general, any tensor A_{ij} with a bar over it has the following meaning (ref. 1):

$$\overline{A}_{ij}\equiv\frac{1}{2}(A_{ij}+A_{ji})-\frac{1}{3}A_{kk}\delta_{ij} \tag{21}$$

Substituting equations (19) and (20) into equations (6) and (7) yields the conventional Navier-Stokes equations

$$\rho\frac{Du_i}{Dt}-\rho F_i+\frac{\partial p}{\partial x_i}-2\frac{\partial}{\partial x_j}\left(\mu\overline{\frac{\partial u_i}{\partial x_j}}\right)=0 \tag{22}$$

and the corresponding energy equation of viscous flow

$$\rho\frac{DE}{Dt}+p\frac{\partial u_i}{\partial x_i}-2\mu\frac{\partial u_i}{\partial x_j}\overline{\frac{\partial u_i}{\partial x_j}}-\frac{\partial}{\partial x_i}\left(\lambda\frac{\partial T}{\partial x_i}\right)=0 \tag{23}$$

From the definition of the nondivergent symmetrical tensor, equation (21),

$$\frac{\partial}{\partial x_j}\left(\mu\overline{\frac{\partial u_i}{\partial x_j}}\right)=\frac{1}{2}\frac{\partial}{\partial x_j}\left(\mu\frac{\partial u_i}{\partial x_j}\right)-\frac{1}{3}\frac{\partial}{\partial x_i}\left(\mu\frac{\partial u_j}{\partial x_j}\right)+\frac{1}{2}\frac{\partial}{\partial x_j}\left(\mu\frac{\partial u_j}{\partial x_i}\right)$$

and

$$\frac{\partial u_i}{\partial x_j}\overline{\frac{\partial u_i}{\partial x_j}}=\overline{\frac{\partial u_i}{\partial x_j}}\,\overline{\frac{\partial u_i}{\partial x_j}}$$

$$=\frac{1}{3}\left[\left(\frac{\partial u_1}{\partial x_1}-\frac{\partial u_2}{\partial x_2}\right)^2+\left(\frac{\partial u_2}{\partial x_2}-\frac{\partial u_3}{\partial x_3}\right)^2+\left(\frac{\partial u_3}{\partial x_3}-\frac{\partial u_1}{\partial x_1}\right)^2\right]+$$

$$\frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right)^2+\left(\frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_2}\right)^2+\left(\frac{\partial u_3}{\partial x_2}+\frac{\partial u_2}{\partial x_3}\right)^2\right]\geqq 0$$

Making use of these relations, it is readily seen that equatons (22) and (23) check with the momentum and energy equations of viscous flow (ref. 10). In the present notation the dissipation function is simply

$$\Phi=2\mu\frac{\partial u_i}{\partial x_j}\overline{\frac{\partial u_i}{\partial x_j}}=2\mu\,\overline{\frac{\partial u_i}{\partial x_j}}\,\overline{\frac{\partial u_i}{\partial x_j}} \tag{24}$$

which is always positive and unaffected in form by the fact that the coefficient of viscosity μ is a variable.

The third-order correction to the molecular-velocity-distribution function, as given by Burnett (ref. 2 or 11), yields the second-order corrections to the stress tensor and the heat-flux vector for both spherical and Maxwell molecules. These corrections are accurate to terms of order $(\mu/p)^2$. In Cartesian tensor notation they are (refs. 1, 3, and 7)[3]

$$\begin{aligned}\tau_{ij}^{(2)}=&K_1\frac{\mu^2}{p}\frac{\partial u_k}{\partial x_k}\overline{\frac{\partial u_i}{\partial x_j}}+\\&K_2\frac{\mu^2}{p}\left[\overline{\frac{\partial}{\partial x_i}\left(F_j-\frac{1}{\rho}\frac{\partial p}{\partial x_j}\right)}-\overline{\frac{\partial u_k}{\partial x_i}\frac{\partial u_j}{\partial x_k}}-2\,\overline{\overline{\frac{\partial u_i}{\partial x_k}}\frac{\partial u_k}{\partial x_j}}\right]+\\&K_3\frac{\mu^2}{\rho T}\overline{\frac{\partial^2 T}{\partial x_i\,\partial x_j}}+K_4\frac{\mu^2}{\rho p T}\overline{\frac{\partial p}{\partial x_i}\frac{\partial T}{\partial x_j}}+\\&K_5\frac{\mu^2}{\rho T^2}\overline{\frac{\partial T}{\partial x_i}\frac{\partial T}{\partial x_j}}+K_6\frac{\mu^2}{p}\overline{\overline{\frac{\partial u_i}{\partial x_k}}\,\overline{\frac{\partial u_k}{\partial x_j}}}\end{aligned} \tag{25}$$

$$\begin{aligned}q_i^{(2)}=&\theta_1\frac{\mu^2}{\rho T}\frac{\partial u_j}{\partial x_j}\frac{\partial T}{\partial x_i}+\\&\theta_2\frac{\mu^2}{\rho T}\left[\frac{2}{3}\frac{\partial}{\partial x_i}\left(T\frac{\partial u_j}{\partial x_j}\right)+2\frac{\partial u_j}{\partial x_i}\frac{\partial T}{\partial x_j}\right]+\\&\left(\theta_3\frac{\mu^2}{\rho p}\frac{\partial p}{\partial x_j}+\theta_4\frac{\mu^2}{\rho}\frac{\partial}{\partial x_j}+\theta_5\frac{\mu^2}{\rho T}\frac{\partial T}{\partial x_j}\right)\overline{\frac{\partial u_j}{\partial x_i}}\end{aligned} \tag{26}$$

The K's and θ's are pure constants. Their correct values are given below (refs. 7 and 9):

Maxwell molecules

$$\left.\begin{aligned}K_1&=\frac{4}{3}\left(\frac{7}{2}-\frac{T}{\mu}\frac{d\mu}{dT}\right)\\K_2&=2\\K_3&=3\\K_4&=0\\K_5&=3\frac{T}{\mu}\frac{d\mu}{dT}\\K_6&=8\end{aligned}\right\} \tag{27}$$

Rigid elastic spherical molecules

$$\left.\begin{aligned}K_1&=\frac{4}{3}\left(\frac{7}{2}-\frac{T}{\mu}\frac{d\mu}{dT}\right)\times 1.014\\K_2&=2\times 1.014\\K_3&=3\times 0.806\\K_4&=0.681\\K_5&=3\frac{T}{\mu}\frac{d\mu}{dT}\times 0.806-0.990\\K_6&=8\times 0.928\end{aligned}\right\} \tag{28}$$

Maxwell molecules

$$\left.\begin{aligned}\theta_1&=\frac{15}{4}\left(\frac{7}{2}-\frac{T}{\mu}\frac{d\mu}{dT}\right)\\\theta_2&=-\frac{45}{8}\\\theta_3&=-3\\\theta_4&=3\\\theta_5&=3\left(\frac{35}{4}+\frac{T}{\mu}\frac{d\mu}{dT}\right)\end{aligned}\right\} \tag{29}$$

Rigid elastic spherical molecules

$$\left.\begin{aligned}\theta_1&=\frac{15}{4}\left(\frac{7}{2}-\frac{T}{\mu}\frac{d\mu}{dT}\right)\times 1.035\\\theta_2&=-\frac{45}{8}\times 1.035\\\theta_3&=-3\times 1.030\\\theta_4&=3\times 0.806\\\theta_5&=3\left(\frac{35}{4}\times 0.918+\frac{T}{\mu}\frac{d\mu}{dT}\times 0.806\right)-0.150\end{aligned}\right\} \tag{30}$$

It is noted that the values of θ_2 and θ_5 for Maxwellian molecules given above are different from those given by Chapman (ref. 2, pp. 267 to 270). These corrections together with the values of θ for rigid elastic spherical molecules are due to Wang Chang and Uhlenbeck (ref. 9). For ordinary gases $(T/\mu)(d\mu/dT)$ has a value lying between 1/2 and 1. It follows from the corrected expression for θ_5 given above that all the coefficients K_i and θ_i are less than 117/4 instead of 45/4 as given by Chapman and Cowling (ref. 2, p. 270).

[3] The last term within the brackets in equation (25) differs from the one given in the references quoted. The correct form for the Burnett terms has been given in reference 9. This error was pointed out to the authors by Prof. S. A. Schaaf and confirmed by Prof. C. A. Truesdell and Mrs. C. S. Wang Chang.

With the expressions for the stress tensor and the heat-flux vector accurate to the second order, the momentum and energy equations, equations (6) and (7), become

$$\rho\frac{Du_i}{Dt}-\rho F_i+\frac{\partial p}{\partial x_i}-2\frac{\partial}{\partial x_j}\left(\mu\overline{\frac{\partial u_i}{\partial x_j}}\right)+\frac{\partial}{\partial x_j}\tau_{ij}^{(2)}=0 \tag{31}$$

$$\rho\frac{DE}{Dt}+p\frac{\partial u_i}{\partial x_i}-2\mu\frac{\partial u_i}{\partial x_j}\overline{\frac{\partial u_i}{\partial x_j}}-\frac{\partial}{\partial x_i}\left(\lambda\frac{\partial T}{\partial x_i}\right)+\tau_{ij}^{(2)}\frac{\partial u_i}{\partial x_j}+\frac{\partial q_i^{(2)}}{\partial x_i}=0 \tag{32}$$

where $\tau_{ij}^{(2)}$ and $q_i^{(2)}$ are given by equations (25) and (26), respectively. Equations (31) and (32) are the momentum and energy equations for slip flow and they reduce to the Navier-Stokes equations and the corresponding energy equation upon neglecting terms containing $\tau_{ij}^{(2)}$ and $q_i^{(2)}$.

From equations (19), (20), (25), and (26) it is seen that the ratio of a typical term of $\tau_{ij}^{(2)}$ to ${}_1\tau_{ij}$ or $q_i^{(2)}$ to ${}_1q_i$ has the same order of magnitude as either $\mu U/pL$ or $\mu^2U/\lambda\rho TL$, where U and L are the characteristic velocity and length of the flow, respectively. Making use of the relations

$$\frac{R}{c_p}=\frac{\gamma-1}{\gamma} \tag{33}$$

and

$$a^2=\gamma RT \tag{34}$$

where R is the gas constant, c_p, the specific heat at constant pressure, γ, the ratio of the specific heats, and a, the adiabatic speed of sound, together with the definitions that the Reynolds number $Re=\rho UL/\mu$, Mach number $M=U/a$, and the Prandtl number $Pr=c_p\mu/\lambda$, it follows that

$$\frac{\mu^2U}{\lambda\rho TL}=(\gamma-1)PrM^2/Re \tag{35}$$

From equation (34) and the perfect-gas law

$$p=R\rho T \tag{36}$$

it also follows that

$$\frac{\mu U}{pL}=\gamma M^2/Re \tag{37}$$

From the kinetic theory of gases (ref. 4, pp. 50 and 147)

$$p/\rho=a^2/\gamma=\pi\bar{c}^2/8 \tag{38}$$

and

$$\mu=0.499\rho\bar{c}l \tag{39}$$

where $\bar{c}$ is the mean molecular speed and l is the mean free path of the gas. From equations (38) and (39)

$$l=\frac{\left(\frac{1}{2}\pi\gamma\right)^{1/2}}{0.998}\frac{\mu}{a\rho}=1.256\gamma^{1/2}\mu/a\rho \tag{40}$$

Hence the Knudsen number is

$$Kn=l/L=1.256\gamma^{1/2}M/Re \tag{41}$$

Substituting equation (41) into equations (37) and (35),

$$\frac{\mu U}{pL}=0.796\gamma^{1/2}MKn \tag{42}$$

$$\frac{\mu^2U}{\lambda\rho TL}=0.796\left(\frac{\gamma-1}{\gamma^{1/2}}\right)PrMKn \tag{43}$$

For ordinary gases the Prandtl number is approximately unity and $\frac{7}{5}\leqq\gamma\leqq\frac{5}{3}$; therefore $\mu U/pL$ and $\mu^2U/\lambda\rho TL$ have the same order of magnitude as $M\ Kn$. Since the rarefaction of the gas increases with the Knudsen number, it is evident that for the high-speed flow of a rarefied gas the second-order terms $\tau_{ij}^{(2)}$ and $q_i^{(2)}$ of the stresses and the heat flux become relatively important. For $\gamma=1.400$ and $Pr=0.750$, it follows from equations (41), (42), and (43) that $Kn=1.486M/Re$, $\mu U/pL=0.940M\ Kn$, and $\mu^2U/\lambda\rho TL=0.201M\ Kn$.

According to Burnett's expression for the molecular-velocity-distribution function, $\tau_{ij}^{(3)}$, the third-order corrections to the stress tensor, will contain terms of the form $(\mu/p)^3(\partial u_i/\partial x_j)^3$, the ratio of which to ${}_1\tau_{ij}$ has the same order of magnitude as

$$\frac{\mu^2}{p^2}\left(\frac{\partial u}{\partial x}\right)^2\approx\left(\frac{\mu U}{pL}\right)^2\approx M^2Kn^2 \tag{44}$$

Similar terms apply for $q_i^{(3)}/{}_1q_i$. Hence the slip-flow equations, equations (31) and (32), cease to be valid if, for a given Mach number, the gas is so rarefied that M^2Kn^2 is not negligible compared with unity.

The particular problem to be considered in this investigation is the slip flow between concentric cylinders. This problem has recently been solved by Schamberg (ref. 3, ch. VII) for the case of constant coefficients of viscosity and thermal conductivity. The present investigation extends Schamberg's solution to include the effect of variable coefficients of viscosity and thermal conductivity.

Assume that the rarefied gas is confined between two concentric cylinders. The inner cylinder, having radius a and the uniform temperature T_{wa}, is rotating at constant angular velocity ω_{wa}, its surface velocity being denoted by $U=a\omega_{wa}$; whereas the outer cylinder, having radius b and the uniform temperature T_{wb}, is held fixed in space. The flow field is conveniently described by the cylindrical polar coordinates r, ϕ, and z, with the z-axis as the axis of the cylinders. Assuming that the flow is two-dimensional and steady and that the external force can be neglected,

$$u_z=0,\ \frac{\partial}{\partial z}=0,\ \frac{\partial}{\partial t}=0,\ F_i=0 \tag{45}$$

It follows from the symmetry of the problem that

$$u_r=0,\ \frac{\partial}{\partial\phi}=0 \tag{46}$$

where u_z and u_r are the velocity components in the z- and r-directions, respectively.

The appropriate equations of motion are obtained by expressing the continuity, momentum and energy equations, equations (1), (31), and (32), in plane polar coordinates.

This is easily done by making use of the formulas in general orthogonal coordinates (refs. 12 and 13). In view of equations (45) and (46) it is found that the equation of continuity is automatically satisfied. The momentum equations in the tangential and radial directions are, respectively (see appendix B),

$$\frac{1}{\rho}\left(\frac{d}{dr}\tau_{r\phi}+\frac{2}{r}\tau_{r\phi}\right)=0 \tag{47}$$

$$-\frac{u_\phi^2}{r}+\frac{1}{\rho}\left[\frac{dp}{dr}+\frac{d}{dr}\tau_{rr}+\frac{1}{r}(\tau_{rr}-\tau_{\phi\phi})\right]=0 \tag{48}$$

The energy equation becomes

$$\frac{d}{dr}q_r+\frac{q_r}{r}+\tau_{r\phi}\left(\frac{du_\phi}{dr}-\frac{u_\phi}{r}\right)=0 \tag{49}$$

In the above equations $\tau_{r\phi}$ is the viscous shearing stress, τ_{rr} and $\tau_{\phi\phi}$ are the normal stresses in the radial and tangential directions, respectively, and q_r is the radial component of the heat-flux vector.

The explicit expressions for the required components of the viscous stress tensor and the heat-flux vector are obtained by transforming the general expressions, equations (25) and (26), respectively, into plane polar coordinates and making the reductions required by equations (45) and (46). This is done in appendix B. The results are given as follows:

$$\tau_{r\phi}=-\mu\left(\frac{du_\phi}{dr}-\frac{u_\phi}{r}\right) \tag{50}$$

$$\begin{aligned}\tau_{rr}=\frac{\mu^2}{p}\Bigg[&\left(\frac{1}{12}K_6-\frac{2}{3}K_2\right)\left(\frac{du_\phi}{dr}\right)^2+\left(\frac{2}{3}K_2-\frac{1}{6}K_6\right)\frac{u_\phi}{r}\frac{du_\phi}{dr}-\\&\frac{2}{3}K_2\frac{d}{dr}\left(\frac{1}{\rho}\frac{dp}{dr}\right)+\left(\frac{1}{12}K_6+\frac{1}{3}K_2\right)\left(\frac{u_\phi}{r}\right)^2+\frac{2}{3}K_4\frac{1}{\rho T}\frac{dp}{dr}\frac{dT}{dr}+\\&\frac{1}{3}\frac{K_2}{r\rho}\frac{dp}{dr}+\frac{2}{3}K_3R\frac{d^2T}{dr^2}+\frac{2}{3}K_5\frac{R}{T}\left(\frac{dT}{dr}\right)^2-\frac{1}{3}K_3R\frac{1}{r}\frac{dT}{dr}\Bigg]\end{aligned} \tag{51}$$

$$\begin{aligned}\tau_{\phi\phi}=\frac{\mu^2}{p}\Bigg\{&\left(\frac{1}{12}K_6+\frac{1}{3}K_2\right)\left(\frac{du_\phi}{dr}\right)^2+\left(\frac{2}{3}K_2-\frac{1}{6}K_6\right)\frac{u_\phi}{r}\frac{du_\phi}{dr}+\\&\left(\frac{1}{12}K_6-\frac{2}{3}K_2\right)\left(\frac{u_\phi}{r}\right)^2-\frac{1}{3}K_4\frac{1}{\rho T}\frac{dp}{dr}\frac{dT}{dr}+\frac{1}{3}K_2\left[\frac{d}{dr}\left(\frac{1}{\rho}\frac{dp}{dr}\right)-\right.\\&\left.\frac{2}{r\rho}\frac{dp}{dr}\right]-\frac{1}{3}K_3R\frac{d^2T}{dr^2}-\frac{1}{3}K_5\frac{R}{T}\left(\frac{dT}{dr}\right)^2+\frac{2}{3}K_3\frac{R}{r}\frac{dT}{dr}\Bigg\}\end{aligned} \tag{52}$$

$$q_r=-\lambda\frac{dT}{dr} \tag{53}$$

The expressions for τ_{rr} and $\tau_{\phi\phi}$ given in equations (51) and (52) are slightly different from those used by Schamberg (ref. 3, pp. 152–153). This is due to the correction of the last term in the brackets of equation (25) for the Burnett terms in the stress tensor. Substituting equations (50), (51), (52), and (53) into equations (47), (48), and (49) and using equation (27) for the values of the K's for Maxwell molecules give

$$\frac{d}{dr}\left[\mu r^3\frac{d}{dr}\left(\frac{u_\phi}{r}\right)\right]=0 \qquad (\text{for } \rho r^2\neq 0) \tag{54}$$

$$\frac{d}{dr}\left(r\lambda\frac{dT}{dr}\right)+\mu r^3\left[\frac{d}{dr}\left(\frac{u_\phi}{r}\right)\right]^2=0 \tag{55}$$

$$\begin{aligned}\frac{dp}{dr}=\frac{\rho u_\phi^2}{r}-\frac{\mu^2}{r p}\Bigg[&-2\frac{d}{dr}\left(\frac{1}{\rho}\frac{dp}{dr}\right)+\frac{2}{r\rho}\frac{dp}{dr}-2\left(\frac{du_\phi}{dr}\right)^2+2\left(\frac{u_\phi}{r}\right)^2+\\&3R\frac{d^2T}{dr^2}-3R\frac{1}{r}\frac{dT}{dr}+3\beta\frac{R}{T}\left(\frac{dT}{dr}\right)^2\Bigg]-\frac{d}{dr}\Bigg\{\left[-\frac{4}{3}\frac{d}{dr}\left(\frac{1}{\rho}\frac{dp}{dr}\right)+\right.\\&\left.\frac{2}{3}\frac{1}{r\rho}\frac{dp}{dr}\right]\frac{\mu^2}{p}+\frac{\mu^2}{p}\left[-\frac{2}{3}\left(\frac{du_\phi}{dr}\right)^2+\frac{4}{3}\left(\frac{u_\phi}{r}\right)^2+2R\frac{d^2T}{dr^2}-R\frac{1}{r}\frac{dT}{dr}+\right.\\&\left.2\beta\frac{R}{T}\left(\frac{dT}{dr}\right)^2\right]\Bigg\}\end{aligned} \tag{56}$$

where $\beta=\frac{T}{\mu}\frac{d\mu}{dT}$. Thus one has three equations, equations (54), (55), and (56), which together with the perfect-gas law, equation (36), are used to find the four dependent variables u_ϕ, T, p, and ρ as functions of r.

BOUNDARY CONDITIONS

It is seen from the previous section that the introduction of the higher order approximations to the stresses and the heat flux results in an increase in the order of the momentum and the energy equations of the fluid mean motion. For instance, the first-order approximations to the stresses ${}_1\tau_{ij}$, equation (19), lead to the conventional Navier-Stokes equations, equations (22), of viscous flow, which are partial differential equations of the second order; while the second-order approximations to the stresses lead to the third-order partial differential equations, equations (31), with the expression for $\tau_{ij}^{(2)}$ given by equation (25). Since the relative importance of the higher order terms of the stresses and the heat flux increases with the rarefaction of the gas, this leads one to the expectation that the number of boundary conditions required for the complete evaluation of a slip-flow problem should likewise depend on the degree of the rarefaction of the gas. However, it was shown by Schamberg (ref. 3) on both physical and mathematical grounds that the number of physical boundary conditions required for a slip-flow problem is effectively the same as that for the corresponding flow in the realm of gas dynamics.

In slip flow, as in gas dynamics, the condition of zero relative normal velocity at the boundary still holds, but the relative tangential velocity at the boundary is no longer zero and the gas temperature differs from the wall temperature. These are known as the "slip velocity" and the "temperature jump," respectively.

The expressions for the slip velocity and the temperature jump at low Mach number were investigated by Maxwell, Millikan, Smoluchowski, Knudsen, and others (refs. 1 and 4). If x and z are the distances tangential and normal to the wall, respectively, u and u_w are, respectively, the velocity of the gas and of the wall in the x-direction, and T and T_w are, respectively, the temperature of the gas and of the wall, one has from the kinetic theory of gases (ref. 4) the slip velocity

$$(u)_{z=0}-u_w=0.998\left(\frac{2-\sigma}{\sigma}\right)\left(\frac{\partial u}{\partial z}\right)_{z=0}l+\frac{3}{4}\frac{\mu}{\rho T}\left(\frac{\partial T}{\partial x}\right)_{z=0} \tag{57}$$

and the temperature jump

$$(T)_{z=0}-T_w=0.998\left(\frac{2}{\gamma+1}\right)\left(\frac{\lambda}{\mu c_v}\right)\left(\frac{2-\alpha}{\alpha}\right)\left(\frac{\partial T}{\partial z}\right)_{z=0}l \tag{58}$$

where σ is Maxwell's reflection coefficient, c_v the specific heat at constant volume, α, the accommodation coefficient, and l, the mean free path.

For slip flow at high Mach number the above expressions give no longer a true description of the physical relations and higher order approximations to the slip velocity and the temperature jump must be used. A general method for the calculation of approximate expressions for the slip velocity and the temperature jump, to an arbitrary degree of approximation, is given by Schamberg (ref. 3). The method applies the laws of conservation of mass, momentum, and energy to the infinitesimal layer of gas adjacent to the solid surface, referred to as the "sublayer," and uses the nonuniform molecular-velocity-distribution function.

The first approximations for the slip velocity and the temperature jump thus obtained by the use of the first approximation to the velocity distribution for a monatomic gas of Maxwell molecules agree with the results given by previous investigators for low-speed slip flows.

The second approximations to the slip velocity and the temperature jump, which are required in conjunction with the second approximations to the viscous stresses and the heat flux for high-speed slip flow, are given as follows (ref. 3): The slip velocity

$$\begin{aligned}u_j(x,y,z)_{z\to 0}-u_{w_j}=&\frac{\mu}{p}\left[a_1(RT)^{1/2}\left(\frac{\partial u_j}{\partial z}+\frac{\partial w}{\partial x_j}\right)+\frac{3}{4}R\frac{\partial T}{\partial x_j}\right]+\left(\frac{\mu}{p}\right)^2\left[-\frac{5}{6}RT\frac{\partial}{\partial z}\left(\frac{\partial u_j}{\partial z}+\frac{\partial w}{\partial x_j}\right)+b_1R\frac{\partial T}{\partial z}\left(\frac{\partial u_j}{\partial z}+\frac{\partial w}{\partial x_j}\right)+\frac{9}{8}R\frac{\partial T}{\partial z}\frac{\partial w}{\partial x_j}-\right.\\&\left.\frac{8}{15}\frac{RT}{\rho}\frac{\partial \rho}{\partial z}\left(\frac{\partial u_j}{\partial z}+\frac{\partial w}{\partial x_j}\right)-3a_1R(RT)^{1/2}\frac{\partial^2 T}{\partial x_j\partial z}-\frac{3}{2}a_1R\frac{(RT)^{1/2}}{\mu}\left(\frac{\partial T}{\partial x_j}\frac{\partial \mu}{\partial z}+\frac{\partial T}{\partial z}\frac{\partial \mu}{\partial x_j}\right)\right]+\left(\frac{\mu}{p}\right)^2\left[-a_1(RT)^{1/2}\frac{D}{Dt}\left(\frac{\partial u_j}{\partial z}+\frac{\partial w}{\partial x_j}\right)+\right.\\&\frac{3}{2}a_1(RT)^{1/2}\frac{1}{T}\frac{DT}{Dt}\left(\frac{\partial u_j}{\partial z}+\frac{\partial w}{\partial x_j}\right)-3a_1(RT)^{1/2}\frac{1}{T}\frac{DT}{Dt}\frac{\partial w}{\partial x_j}-a_1(RT)^{1/2}\left(\frac{\partial u_j}{\partial z}+\frac{\partial w}{\partial x_j}\right)\frac{D}{Dt}\left(\log_e\frac{\mu}{p}\right)-\\&\left.\frac{5}{8}R\frac{D}{Dt}\left(\frac{\partial T}{\partial x_j}\right)+b_2\frac{R}{T}\frac{DT}{Dt}\frac{\partial T}{\partial x_j}+\frac{9}{8}\frac{R}{\rho}\frac{D\rho}{Dt}\frac{\partial T}{\partial x_j}+\frac{4}{5}\frac{R}{\rho}\frac{DT}{Dt}\frac{\partial \rho}{\partial x_j}-\frac{5}{6}\left(\frac{\partial u_j}{\partial z}+\frac{\partial w}{\partial x_j}\right)\frac{Dw}{Dt}\right]\end{aligned} \tag{59}$$

where $j=1$ and 2, and the temperature jump

$$\begin{aligned}T(x,y,z)_{z\to 0}-T_w=&\frac{\mu}{p}\left[c_1(RT)^{1/2}\frac{\partial T}{\partial z}-\frac{1}{2}\frac{DT}{Dt}\right]+\left(\frac{\mu}{p}\right)^2\left[e_1T\left(\frac{\partial u_j}{\partial z}+\frac{\partial w}{\partial x_j}\right)^2-\frac{1}{2}T\frac{\partial w}{\partial x_j}\left(\frac{\partial u_j}{\partial z}+\frac{\partial w}{\partial x_j}\right)+e_2T(RT)^{1/2}\frac{\partial}{\partial x_j}\left(\frac{\partial u_j}{\partial z}+\frac{\partial w}{\partial x_j}\right)+\right.\\&e_3T(RT)^{1/2}\left(2\frac{\partial^2 w}{\partial z^2}+\frac{\partial^2 w}{\partial x_j\partial x_j}\right)+e_4(RT)^{1/2}\frac{\partial T}{\partial x_j}\left(\frac{\partial u_j}{\partial z}+\frac{\partial w}{\partial x_j}\right)+e_5(RT)^{1/2}\left(\frac{\partial T}{\partial x_j}\frac{\partial w}{\partial x_j}\right)+\frac{1}{6}a_1(RT)^{1/2}\frac{T}{\rho}\frac{\partial \rho}{\partial x_j}\left(\frac{\partial u_j}{\partial z}+\frac{\partial w}{\partial x_j}\right)-\\&\frac{1}{2}a_1(RT)^{1/2}\frac{T}{\mu}\frac{\partial \mu}{\partial x_j}\left(\frac{\partial u_j}{\partial z}+\frac{\partial w}{\partial x_j}\right)+e_6RT\frac{\partial^2 T}{\partial x_j\partial x_j}+e_7R\frac{\partial T}{\partial x_j}\frac{\partial T}{\partial x_j}+e_8R\left(\frac{\partial T}{\partial z}\right)^2+\frac{1}{8}\frac{RT}{\rho}\frac{\partial T}{\partial x_j}\frac{\partial \rho}{\partial x_j}+e_6\frac{RT}{\mu}\frac{\partial \mu}{\partial x_j}\frac{\partial T}{\partial x_j}-\\&\left.\frac{1}{14}RT\frac{\partial^2 T}{\partial z^2}-\frac{1}{14}\frac{RT}{\mu}\frac{\partial T}{\partial z}\frac{\partial \mu}{\partial z}\right]+\left(\frac{\mu}{p}\right)^2\left\{e_9\frac{1}{T}\left(\frac{DT}{Dt}\right)^2+\frac{1}{2}\frac{DT}{Dt}\frac{D}{Dt}\left(\log_e\frac{\mu}{p}\right)+e_{10}(RT)^{1/2}\frac{1}{T}\frac{DT}{Dt}\frac{\partial T}{\partial z}+\right.\\&\left.e_5(RT)^{1/2}\frac{D}{Dt}\left(\frac{\partial T}{\partial z}\right)-e_5(RT)^{1/2}\frac{\partial T}{\partial z}\frac{D}{Dt}[\log_e(\rho T^{1/2})]+\frac{1}{2}\frac{D^2T}{Dt^2}-3e_3\frac{1}{(RT)^{1/2}}\frac{DT}{Dt}\frac{Dw}{Dt}-\frac{18}{7}\frac{\partial T}{\partial z}\frac{Dw}{Dt}\right\}\end{aligned} \tag{60}$$

where $j=1$ and 2.

In equation (60) but not in equation (59) the summation convention over two indices is used. This notation has the meaning:

$$\left.\begin{aligned}x_1\equiv x, x_2\equiv y\\u_1\equiv u, u_2\equiv v\end{aligned}\right\} \tag{61}$$

so that

$$\frac{\partial h}{\partial x_j}\left(\frac{\partial u_j}{\partial z}+\frac{\partial w}{\partial x_j}\right)=\frac{\partial h}{\partial x}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\frac{\partial h}{\partial y}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \tag{62}$$

The time derivative D/Dt, when expressed in the above notation, is

$$\frac{D}{Dt}\equiv\frac{\partial}{\partial t}+u_{w_j}\frac{\partial}{\partial x_j} \tag{63}$$

All of the derivatives in equations (59) and (60) are to be evaluated at a point (x, y, z) as $z\to 0$. These boundary conditions are applicable provided that the pressure level and the motion of the gas are such that the second approximations to the stress tensor ${}_2\tau_{ij}$ and the heat-flux vector ${}_2q_i$ are applicable. This means that the relation $M^2Kn^2\ll 1$ holds.

The values of a_1, the b's, c_1, and the e's in equations (59) and (60) are given as follows (ref. 3):[4]

$$
\left.
\begin{aligned}
a_1&=\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{2-\sigma}{\sigma}\right)\\
b_1&=-5.167\\
b_2&=0.8749\\
c_1&=\frac{15}{8}\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{2-\alpha}{\alpha}\right)\\
c_1'&=\left(\frac{2}{\gamma+1}\right)\left(\frac{\lambda}{\mu c_v}\right)\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{2-\alpha}{\alpha}\right)\\
e_1&=-\left[0.31655+\frac{\pi}{8}\left(\frac{2-\sigma}{\sigma}\right)^2-\frac{\pi}{4}\left(\frac{2-\sigma}{\sigma}\right)\left(\frac{2-\alpha}{\alpha}\right)\right]\\
e_2&=-\left(\frac{\pi}{2}\right)^{1/2}\left[\frac{1}{4}\left(\frac{2-\alpha}{\alpha}\right)+\frac{1}{2}\left(\frac{2-\sigma}{\sigma}\right)\right]\\
e_3&=-\frac{1}{2}\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{2-\alpha}{\alpha}\right)\\
e_4&=-\left(\frac{\pi}{2}\right)^{1/2}\left[\frac{33}{8}\left(\frac{2-\alpha}{\alpha}\right)-\frac{1}{4}\left(\frac{2-\sigma}{\sigma}\right)\right]\\
e_5&=-\frac{45}{16}\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{2-\alpha}{\alpha}\right)\\
e_6&=\frac{107}{56}\\
e_7&=-7.9888\\
e_8&=-5.4912\\
e_9&=-1.7183\\
e_{10}&=\frac{159}{16}\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{2-\alpha}{\alpha}\right)
\end{aligned}
\right\}
\tag{64}
$$

In general c_1' is used instead of c_1. According to the kinetic theory for monatomic gases (ref. 2) $\gamma=5/3$, $\lambda/\mu c_v=5/2$, and c_1' reduces to c_1. The values of σ for air vary from 0.79 to 1.00 while α for air lies between 0.88 and 0.97 (ref. 1).

To obtain the boundary conditions for the concentric-cylinder flow, it is first necessary to express the general boundary conditions, as given by equations (59) and (60) in the Cartesian coordinate system, in terms of the polar coordinate system. Following Schamberg (ref. 3) closely, let x and y be the Cartesian axes associated with the polar coordinates r and ϕ and the auxiliary coordinate systems x_a, z_a and x_b, z_b by the equations

$$
\left.
\begin{aligned}
x&=a+z_a=r\cos\phi\\
y&=x_a=r\sin\phi
\end{aligned}
\right\}
\tag{65}
$$

$$
\left.
\begin{aligned}
x&=b-z_b=r\cos\phi\\
y&=-x_b=r\sin\phi
\end{aligned}
\right\}
\tag{66}
$$

The velocity components u_a, w_a and u_b, w_b of the auxiliary coordinate systems are related to the tangential velocity u_ϕ as shown by figure 1 and equations (67) and (68):

$$
\left.
\begin{aligned}
u_a&=u_\phi\cos\phi\\
w_a&=-u_\phi\sin\phi
\end{aligned}
\right\}
\tag{67}
$$

$$
\left.
\begin{aligned}
u_b&=-u_\phi\cos\phi\\
w_b&=u_\phi\sin\phi
\end{aligned}
\right\}
\tag{68}
$$

The partial derivatives with respect to the auxiliary coordinates are expressed in terms of the partial derivatives with respect to r and ϕ by means of equations (69) and (70), which are easily obtained from equations (65) and (66), respectively:

$$
\left.
\begin{aligned}
\frac{\partial}{\partial x_a}&=\sin\phi\frac{\partial}{\partial r}+\cos\phi\frac{1}{r}\frac{\partial}{\partial\phi}\\
\frac{\partial}{\partial z_a}&=\cos\phi\frac{\partial}{\partial r}-\sin\phi\frac{1}{r}\frac{\partial}{\partial\phi}
\end{aligned}
\right\}
\tag{69}
$$

$$
\left.
\begin{aligned}
\frac{\partial}{\partial x_b}&=-\sin\phi\frac{\partial}{\partial r}-\cos\phi\frac{1}{r}\frac{\partial}{\partial\phi}\\
\frac{\partial}{\partial z_b}&=-\cos\phi\frac{\partial}{\partial r}+\sin\phi\frac{1}{r}\frac{\partial}{\partial\phi}
\end{aligned}
\right\}
\tag{70}
$$

All of the first- and second-order partial derivatives appearing in equations (59) and (60) for the boundary conditions can now be transformed into polar coordinates by means of equations (67) to (70). After all of the differentiations with respect to ϕ have been performed, ϕ is put equal to zero in the resultant expressions, in accordance with figure 1. Because of the condition of axial symmetry, equation (46), the partial derivatives of u_ϕ, T, p, and ρ with respect to ϕ all vanish.

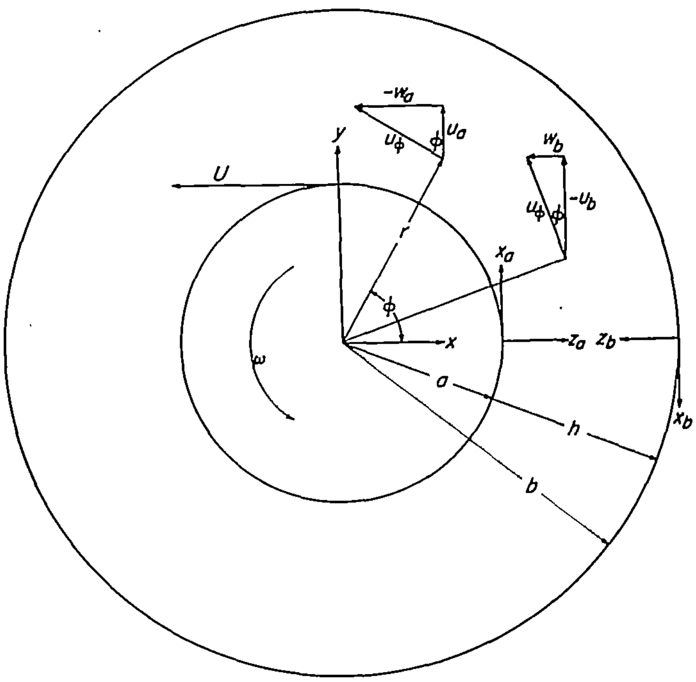


FIGURE 1.—Coordinate systems.

The boundary-condition derivatives for both the convex ($r=a$) and the concave ($r=b$) cylindrical surfaces are given in the following table. The transformations for the deriva-

[4] Schamberg's values for these constants have not been changed to agree with the corrected values of the K's and θ's given by equations (27) and (29) because the corrections are negligible to the order of approximation used later on.

tive D/Dt are obtained from equations (63), (45), and (46), using the fact that the velocity of the wall u_w has the value U at $r=a$ and zero at $r=b$, respectively.

Cartesian derivative	Equivalent radial derivative at—	
	Convex surface ($r=a$)	Concave surface ($r=b$)
$\frac{\partial T}{\partial x}$	0	0
$\frac{\partial T}{\partial z}$	$\frac{dT}{dr}$	$-\frac{dT}{dr}$
$\frac{\partial^2 T}{\partial x^2}$	$\frac{1}{r}\frac{dT}{dr}$	$\frac{1}{r}\frac{dT}{dr}$
$\frac{\partial^2 T}{\partial z^2}$	$\frac{d^2T}{dr^2}$	$\frac{d^2T}{dr^2}$
$\frac{\partial^2 T}{\partial x\,\partial z}$	0	0
$\frac{\partial u}{\partial x}$	0	0
$\frac{\partial u}{\partial z}$	$\frac{du_\phi}{dr}$	$\frac{du_\phi}{dr}$
$\frac{\partial^2 u}{\partial x^2}$	$\frac{d}{dr}\left(\frac{u_\phi}{r}\right)$	$-\frac{d}{dr}\left(\frac{u_\phi}{r}\right)$
$\frac{\partial^2 u}{\partial z^2}$	$\frac{d^2u_\phi}{dr^2}$	$-\frac{d^2u_\phi}{dr^2}$
$\frac{\partial^2 u}{\partial x\,\partial z}$	0	0
$\frac{\partial w}{\partial x}$	$-\frac{u_\phi}{r}$	$-\frac{u_\phi}{r}$
$\frac{\partial w}{\partial z}$	0	0
$\frac{\partial^2 w}{\partial x^2}$	0	0
$\frac{\partial^2 w}{\partial z^2}$	0	0
$\frac{\partial^2 w}{\partial x\,\partial z}$	$-\frac{d}{dr}\left(\frac{u_\phi}{r}\right)$	$\frac{d}{dr}\left(\frac{u_\phi}{r}\right)$
u	u_ϕ	$-u_\phi$
$\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}$	$r\frac{d}{dr}\left(\frac{u_\phi}{r}\right)$	$r\frac{d}{dr}\left(\frac{u_\phi}{r}\right)$
$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)$	$\frac{d}{dr}\left[r\frac{d}{dr}\left(\frac{u_\phi}{r}\right)\right]$	$-\frac{d}{dr}\left[r\frac{d}{dr}\left(\frac{u_\phi}{r}\right)\right]$
$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)$	0	0
$\frac{DT}{Dt}$	0	0
$\frac{D^2T}{Dt^2}$	$\left(\frac{U^2}{r}\right)\left(\frac{dT}{dr}\right)$	0
$\frac{D}{Dt}\left(\frac{\partial T}{\partial x}\right)$	$\frac{U}{r}\frac{dT}{dr}$	0
$\frac{D}{Dt}\left(\frac{\partial T}{\partial z}\right)$	0	0
$\frac{D}{Dt}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)$	0	0
$\frac{Dw}{Dt}$	$-\frac{Uu_\phi}{r}$	0

With these substitutions one obtains from equations (59) and (60) the following four boundary conditions:

(1) At $r=a$,

$$(u_\phi)_a=U+a_1(RT_a)^{1/2}\frac{\mu_a}{p_a}\left[r\frac{d}{dr}\left(\frac{u_\phi}{r}\right)\right]_a+\frac{\mu_a^2}{p_a^2}X_a \tag{71}$$

(2) At $r=b$,

$$(u_\phi)_b=0-a_1\,(RT_b)^{1/2}\frac{\mu_b}{p_b}\left[r\frac{d}{dr}\left(\frac{u_\phi}{r}\right)\right]_b+\frac{\mu_b^2}{p_b^2}X_b \tag{72}$$

(3) At $r=a$,

$$T_a=T_{wa}+c_1(RT_a)^{1/2}\frac{\mu_a}{p_a}\left(\frac{dT}{dr}\right)_a+\frac{\mu_a^2}{p_a^2}Z_a \tag{73}$$

(4) At $r=b$,

$$T_b=T_{wb}-c_1(RT_b)^{1/2}\frac{\mu_b}{p_b}\left(\frac{dT}{dr}\right)_b+\frac{\mu_b^2}{p_b^2}Z_b \tag{74}$$

The subscripts a and b denote the evaluation of a particular quantity at $r=a$ and $r=b$, respectively. The quantities X_a, X_b, Z_a, and Z_b are defined by the following equations:

$$\begin{aligned}X_a=&-\frac{5}{6}RT_a\left\{\frac{d}{dr}\left[r\frac{d}{dr}\left(\frac{u_\phi}{r}\right)\right]\right\}_a-5.167R\left(\frac{dT}{dr}\right)_a\left[r\frac{d}{dr}\left(\frac{u_\phi}{r}\right)\right]_a-\\&\frac{9}{8}R\left(\frac{dT}{dr}\frac{u_\phi}{r}\right)_a-\frac{8}{15}RT_a\left[\frac{1}{p}\frac{dp}{dr}r\frac{d}{dr}\left(\frac{u_\phi}{r}\right)\right]_a+\\&\frac{8}{15}R\left[\frac{dT}{dr}r\frac{d}{dr}\left(\frac{u_\phi}{r}\right)\right]_a-\frac{5}{8}RU\left(\frac{1}{r}\frac{dT}{dr}\right)_a+\frac{5}{6}U\left[u_\phi\frac{d}{dr}\left(\frac{u_\phi}{r}\right)\right]_a\end{aligned} \tag{75}$$

$$\begin{aligned}X_b=&-\frac{5}{6}RT_b\left\{\frac{d}{dr}\left[r\frac{d}{dr}\left(\frac{u_\phi}{r}\right)\right]\right\}_b-5.167R\left(\frac{dT}{dr}\right)_b\left[r\frac{d}{dr}\left(\frac{u_\phi}{r}\right)\right]_b-\\&\frac{9}{8}R\left(\frac{dT}{dr}\frac{u_\phi}{r}\right)_b-\frac{8}{15}RT_b\left[\frac{1}{p}\frac{dp}{dr}r\frac{d}{dr}\left(\frac{u_\phi}{r}\right)\right]_b+\\&\frac{8}{15}R\left[\frac{dT}{dr}r\frac{d}{dr}\left(\frac{u_\phi}{r}\right)\right]_b\end{aligned} \tag{76}$$

$$\begin{aligned}Z_a=T_a\Bigg\{&e_1\left[r\frac{d}{dr}\left(\frac{u_\phi}{r}\right)\right]_a^2+\frac{1}{2}\left[u_\phi\frac{d}{dr}\left(\frac{u_\phi}{r}\right)\right]_a+e_6R\left(\frac{1}{r}\frac{dT}{dr}\right)_a-\\&\frac{1}{14}\frac{\beta R}{T_a}\left(\frac{dT}{dr}\right)_a^2+e_8\frac{R}{T_a}\left(\frac{dT}{dr}\right)_a^2-\frac{1}{14}R\left(\frac{d^2T}{dr^2}\right)_a+\\&\frac{U^2}{T_a}\left[\left(\frac{1}{2}+\frac{18}{7}\frac{u_\phi}{U}\right)\frac{1}{r}\frac{dT}{dr}\right]_a\Bigg\}\end{aligned} \tag{77}$$

$$\begin{aligned}Z_b=T_b\Bigg\{&e_1\left[r\frac{d}{dr}\left(\frac{u_\phi}{r}\right)\right]_b^2+\frac{1}{2}\left[u_\phi\frac{d}{dr}\left(\frac{u_\phi}{r}\right)\right]_b+e_6R\left(\frac{1}{r}\frac{dT}{dr}\right)_b+\\&e_8\frac{R}{T_b}\left(\frac{dT}{dr}\right)_b^2-\frac{1}{14}R\left(\frac{d^2T}{dr^2}\right)_b-\frac{1}{14}\frac{\beta R}{T_b}\left(\frac{dT}{dr}\right)_b^2\Bigg\}\end{aligned} \tag{78}$$

The values of the numerical constants a_1, c_1, e_1, e_6, and e_8 are given by equations (64).

SOLUTION OF CONCENTRIC-CYLINDER FLOW

From the preceding sections it is seen that the problem of slip flow between concentric cylinders is reduced to that of solving the three differential equations, equations (54), (55), and (56), and satisfying the four boundary conditions given by equations (71) to (74). For this purpose it is desirable to introduce dimensionless constants and variables as follows:

$$k=a/b \tag{79}$$

$$\left.\begin{aligned} M_{wa}^2&=\frac{U^2}{\gamma RT_{wa}} \\ Re_{wa}&=\frac{\rho_a Uh}{\mu_{wa}} \\ Pr_{wa}&=\frac{\mu_{wa}c_{p_{wa}}}{\lambda_{wa}} \end{aligned}\right\} \tag{80}$$

$$\left.\begin{matrix} r^*=r/a & p^*=p/p_a & \lambda^*=\lambda/\lambda_{wa} \\ u^*=u_\phi/U & \rho^*=\rho/\rho_a & c_p^*=c_p/c_{p_{wa}} \\ T^*=T/T_{wa} & \mu^*=\mu/\mu_{wa} & Pr^*=Pr/Pr_{wa} \end{matrix}\right\} \tag{81}$$

where

$$h=b-a \tag{82}$$

and $c_{p_{wa}}$, μ_{wa}, and λ_{wa} are the properties of the gas based upon the wall temperature T_{wa}. Making use of the perfect-gas law, equation (36), and of equations (79) to (82), equations (54), (55), (56), and (36) become, respectively,

$$\frac{d}{dr^*}\left[(r^*)^3\mu^*\frac{d}{dr^*}\left(\frac{u^*}{r^*}\right)\right]=0 \tag{83}$$

$$\frac{d}{dr^*}\left(r^*\lambda^*\frac{dT^*}{dr^*}\right)+(\gamma-1)Pr_{wa}M_{wa}^2(r^*)^3\mu^*\left[\frac{d}{dr^*}\left(\frac{u^*}{r^*}\right)\right]^2=0 \tag{84}$$

$$\begin{aligned}\frac{d\log_e p^*}{dr^*}&=\frac{1}{p^*}\frac{dp^*}{dr^*} \\ &=\gamma M_{wa}^2\frac{(u^*)^2}{r^*T^*}-\frac{(1-k)^2\gamma M_{wa}^2}{k^2(T_a^*)^2Re_{wa}^2}\left\{\frac{(\mu^*)^2}{r^*(p^*)^2}\left[-2\gamma M_{wa}^2\left(\frac{du^*}{dr^*}\right)^2+\right.\right. \\ &\quad 2\gamma M_{wa}^2\left(\frac{u^*}{r^*}\right)^2+3\frac{d^2T^*}{(dr^*)^2}-3\frac{1}{r^*}\frac{dT^*}{dr^*}+\frac{3\beta}{T^*}\left(\frac{dT^*}{dr^*}\right)^2- \\ &\quad \left.\left.2\frac{d}{dr^*}\left(T^*\frac{d\log_e p^*}{dr^*}\right)+2\frac{T^*}{r^*}\frac{d\log_e p^*}{dr^*}\right]\right\}- \\ &\quad \frac{(1-k)^2\gamma M_{wa}^2}{k^2p^*(T_a^*)^2Re_{wa}^2}\frac{d}{dr^*}\left\{\frac{(\mu^*)^2}{p^*}\left[-\frac{2}{3}\gamma M_{wa}^2\left(\frac{du^*}{dr^*}\right)^2+\right.\right. \\ &\quad \frac{4}{3}\gamma M_{wa}^2\left(\frac{u^*}{r^*}\right)^2+2\frac{d^2T^*}{(dr^*)^2}-\frac{1}{r^*}\frac{dT^*}{dr^*}+\frac{2\beta}{T^*}\left(\frac{dT^*}{dr^*}\right)^2- \\ &\quad \left.\left.\frac{4}{3}\frac{d}{dr^*}\left(T^*\frac{d\log_e p^*}{dr^*}\right)+\frac{2}{3}\frac{T^*}{r^*}\frac{d\log_e p^*}{dr^*}\right]\right\}\end{aligned} \tag{85}$$

$$p^*T_a^*=\rho^*T^* \tag{86}$$

VELOCITY AND TEMPERATURE DISTRIBUTION

Integration of equation (83) gives

$$u^*=Ar^*-2Br^*\int\frac{dr^*}{(r^*)^3\mu^*} \tag{87}$$

where A and $2B$ are constants of integration. For μ=Constant, $\mu^*=1$ and equation (87) yields

$$u^*=Ar^*+\frac{B}{r^*} \tag{87a}$$

which is Schamberg's solution.

From equations (84) and (87)

$$(r^*)^3\mu^*\frac{d}{dr^*}\left(r^*\lambda^*\frac{dT^*}{dr^*}\right)+4(\gamma-1)Pr_{wa}M_{wa}^2B^2=0 \tag{88}$$

It is convenient to transform the independent variable r^* into ξ or ζ. The latter are defined, respectively, as

$$\xi=\frac{\log_e r^*}{\log_e 1/k}=\frac{1}{m}\log_e r^* \qquad m=\log_e 1/k \tag{89}$$

and

$$\zeta=\eta\int_0^\xi\frac{d\xi}{\mu^*} \tag{90}$$

where

$$\frac{1}{\eta}=\int_0^1\frac{d\xi}{\mu^*} \tag{91}$$

Equations (89) to (91) and (79) insure that

$$\xi=\zeta=0 \text{ at } r^*=1 \text{ or } r=a$$

$$\xi=\zeta=1 \text{ at } r^*=1/k \text{ or } r=b$$

In terms of ξ equation (88) becomes

$$\mu^*\frac{d}{d\xi}\left(\lambda^*\frac{dT^*}{d\xi}\right)+4(\gamma-1)Pr_{wa}M_{wa}^2m^2B^2k^{2\xi}=0 \tag{92}$$

while in terms of the independent variable ζ it is

$$\frac{d}{d\zeta}\left(\frac{\lambda^*}{\mu^*}\frac{dT^*}{d\zeta}\right)+\frac{4(\gamma-1)}{\eta^2}\,Pr_{wa}M_{wa}^2m^2B^2(k^2)^{\zeta+\Delta(\zeta)}=0$$

if

$$\Delta(\zeta)=\int_0^\zeta\left(\frac{\mu^*}{\eta}-1\right)d\zeta=\xi-\zeta \tag{93}$$

Equation (93) together with equations (90) and (91) gives $\Delta(0)=\Delta(1)=0$. In general $\Delta(\zeta)\ll1$.

Integrating once

$$\frac{\lambda^*}{\mu^*}\frac{dT^*}{d\zeta}=mC-\frac{4(\gamma-1)}{\eta^2}\,Pr_{wa}M_{wa}^2m^2B^2\int(k^2)^{\zeta+\Delta(\zeta)}d\zeta$$

and again

$$T^*=D+mC\int\frac{\mu^*}{\lambda^*}\,d\zeta-\frac{4(\gamma-1)}{\eta^2}\,Pr_{wa}M_{wa}^2m^2B^2\int\frac{\mu^*}{\lambda^*}\,d\zeta\times\int(k^2)^{\zeta+\Delta(\zeta)}\,d\zeta \tag{94}$$

For constant μ and λ, $\mu^*=\lambda^*=1$, $\eta=1$, $\zeta=\xi$, and equation (94) reduces to

$$T^*=D+C\log_e r^*-(\gamma-1)Pr_{wa}M_{wa}^2B^2(r^*)^{-2}$$

which is Schamberg's solution.

In terms of ζ, equation (87) becomes

$$\omega^*\equiv\frac{u^*}{r^*}=A-\frac{2Bm}{\eta}\int(k^2)^{\zeta+\Delta(\zeta)}\,d\zeta \tag{95}$$

Now let

$$(k^2)^{\Delta(\zeta)}=1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\,[2m\Delta(\zeta)]^n \tag{96}$$

This series converges for all finite values of $m\Delta(\zeta)$. Making use of equation (96), equation (95) gives

$$\omega^*=A+\frac{B}{\eta}\,[k^{2\zeta}+f(\zeta)] \tag{97}$$

where

$$f(\zeta)=\sum_{n=1}^{\infty}\frac{(-2m)^{n+1}}{n!}\int k^{2\zeta}\,[\Delta(\zeta)]^n\,d\zeta \tag{98}$$

Similarly, from equations (96) and (94), assuming $Pr/c_p=$ Constant,

$$T^*=D+mC\zeta+\frac{\gamma-1}{\eta^2}\,Pr_{wa}M_{wa}^2B^2\,[-k^{2\zeta}+g(\zeta)] \tag{99}$$

where

$$g(\zeta)=2m\int f(\zeta)d\zeta \tag{100}$$

Now at $r=a$, $\zeta=0$, $\omega^*=\omega_a^*$, and $T^*=T_a^*$, and, at $r=b$, $\zeta=1$, $\omega^*=\omega_b^*$, and $T^*=T_b^*$. Therefore, from equations (97) and (99)

$$\omega^*=\omega_a^*-(\omega_a^*-\omega_b^*)\,\frac{1-k^{2\zeta}-F(\zeta)}{1-k^2-F(1)} \tag{101}$$

where

$$F(\zeta)=f(\zeta)-f(0) \tag{102}$$

and

$$T^*=T_a^*-(T_a^*-T_b^*)\zeta+\frac{(\gamma-1)Pr_{wa}M_{wa}^2(\omega_a^*-\omega_b^*)^2}{[1-k^2-F(1)]^2}\,[1-\zeta+k^2\zeta-k^{2\zeta}+G(\zeta)] \tag{103}$$

where

$$G(\zeta)=g(\zeta)-g(0)-[g(1)-g(0)]\zeta \tag{104}$$

It is noted that $F(0)=G(0)=G(1)=0$.

Equations (101) and (103) represent the exact solutions of differential equations (83) and (84). Since the latter did not contain any Burnett terms, it was relatively simple to carry out the integration. However, $\Delta(\zeta)$ depends on a knowledge of η and μ^* which are functions of T^* and so any numerical solution will reduce to an iteration process of approximating $\Delta(\zeta)$ and hence $F(\zeta)$ and $G(\zeta)$. This is also the method of determining the pressure distribution p^* from equation (85). Before proceeding to p^*, it is advantageous to rewrite the boundary conditions of the section "Boundary Conditions" in a similar dimensionless form since the constants ω_a^*, ω_b^*, T_a^*, and T_b^* in equations (101) and (103) are precisely these boundary values. The actual algebraic reduction is carried out in appendix C, leading to the following results:

$$\omega_a^*=1-1.592a_1\,\frac{\omega_a^*-\omega_b^*}{(T_a^*)^{1/2}}\,\frac{1-k}{k[1-k^2-F(1)]}\,Kn_e+\tilde{X}_aKn_e^2 \tag{105}$$

$$\omega_b^*=1.592a_1\,\frac{(\omega_a^*-\omega_b^*)(T_b^*)^{1/2}}{T_a^*p_b^*}\,\frac{k^2(1-k)}{1-k^2-F(1)}\,Kn_e+k\tilde{X}_bKn_e^2 \tag{106}$$

$$T_a^*=1-0.796c_1\,\frac{1-k}{km\,(T_a^*)^{1/2}}\left\{T_a^*-T_b^*+(\gamma-1)\,Pr_{wa}M_{wa}^2(\omega_a^*-\omega_b^*)^2\,\frac{1-k^2-2m-G'(0)}{[1-k^2-F(1)]^2}\right\}Kn_e+\tilde{Z}_aKn_e^2 \tag{107}$$

$$T_b^*=T_{wb}^*+0.796c_1\,\frac{(1-k)\,(T_b^*)^{1/2}}{mT_a^*p_b^*}\left\{T_a^*-T_b^*+(\gamma-1)Pr_{wa}\,M_{wa}^2\,(\omega_a^*-\omega_b^*)^2\,\frac{1-k^2-2k^2m-G'(1)}{[1-k^2-F(1)]^2}\right\}Kn_e+\tilde{Z}_bKn_e^2 \tag{108}$$

where

$$Kn_e=\eta\,Kn_{wa} \tag{109}$$

and $\tilde{X}_a$, $\tilde{X}_b$, $\tilde{Z}_a$, and $\tilde{Z}_b$ are the complicated expressions given in equation (C12). Actually, since $\tilde{X}_a$ . . . $\tilde{Z}_b$ are multiplied by Kn_e^2 above, it is only necessary to use the zero-order approximation to these expressions given by equations

(C32), (C33), and (C34). The reason for this, of course, is that the differential equations and the boundary conditions have been derived only to terms in the square of the Knudsen number. Thus, it is probably clearer to write equations (105) to (108) in the form (eqs. (C27))

$$\omega_a^* = {}_0\omega_a^*(1 + {}_1\omega_a^* Kn_e + {}_2\omega_a^* Kn_e^2)$$

$$\omega_b^* = {}_0\omega_b^* + {}_1\omega_b^* Kn_e + {}_2\omega_b^* Kn_e^2$$

$$T_a^* = {}_0T_a^*(1 + {}_1T_a^* Kn_e + {}_2T_a^* Kn_e^2)$$

$$T_b^* = {}_0T_b^*(1 + {}_1T_b^* Kn_e + {}_2T_b^* Kn_e^2)$$

which explicitly indicates an expansion in powers of Kn_e. The coefficients ${}_h\omega_a^*$, ${}_h\omega_b^*$, ${}_hT_a^*$, and ${}_hT_b^*$ ($h=0$, 1, 2) are written out in equations (C29) to (C31) of appendix C. The subscript written in front of a symbol thus denotes the order of the approximation in the expansion.

PRESSURE DISTRIBUTION

From equation (85), neglecting the terms containing $\left(\frac{M_{wa}}{Re_{wa}}\right)^2$, which are the order of $\bar{\omega}^2$ or Kn^2, the zero-order approximation to p^* is

$$^0p^* = \exp\left[\gamma M_{wa}^2 \int_1^{r^*} \frac{(u^*)^2\,dr^*}{r^* T^*}\right] \tag{110}$$

The superscript in front of a symbol denotes the order of the approximation to the solution of a differential equation.

From equations (89) and (90)

$$\frac{dr^*}{r^*} = \frac{m}{\eta}\,\mu^* d\zeta$$

Therefore

$$^0p^* = \exp\left[\frac{\gamma m}{\eta} M_{wa}^2 \int_0^{\zeta} \frac{\mu^*}{T^*}(u^*)^2 d\zeta\right] \tag{111}$$

or

$$^0p^* = \exp\left[\frac{\gamma m}{\eta} M_{wa}^2 \int_0^{\zeta} \frac{\mu^*(\omega^*)^2}{T^*(k^2)^{\zeta+\Delta(\zeta)}}\,d\zeta\right] \tag{112}$$

Since equation (85) does not explicitly contain terms of order Kn, the zero- and the first-order approximation for p^* can be obtained from equation (110), (111), or (112). From equations (112) and (C29)

$$\left.\begin{aligned} {}_0\left(\frac{d\log_e p^*}{d\zeta}\right)_0 &= \gamma M_{wa}^2 \frac{m}{{}_0\eta}\,{}_0\mu^* \\ {}_0\left(\frac{d\log_e p^*}{d\zeta}\right)_1 &= 0 \end{aligned}\right\} \tag{113}$$

From equations (85), (C5), and (110), $\log_e {}^2p^* = \log_e {}^0p^* + {}_0\psi Kn_{wa}^2$ or

$$^2p^* = (1 + {}_0\psi Kn_{wa}^2)\exp\left[\frac{\gamma m}{\eta} M_{wa}^2 \int_0^{\zeta} \frac{\mu^*(\omega^*)^2\,d\zeta}{T^*(k^2)^{\zeta+\Delta(\zeta)}}\right] \tag{114}$$

with $\exp({}_0\psi Kn_{wa}^2) \approx 1 + {}_0\psi\, Kn_{wa}^2$, where

$$\begin{aligned} {}_0\psi = &-\frac{0.6336(1-k)^2}{k^2}\int_1^{r^*} \frac{{}_0(\mu^*)^2\,dr^*}{r^*\,{}_0(p^*)^2}\Bigg\{2\gamma M_{wa}^2\,{}_0\left(\frac{u^*}{r^*} - \frac{du^*}{dr^*}\right)_0\left(3\frac{u^*}{r^*} + \frac{du^*}{dr^*}\right) + 3\,{}_0\left[\frac{d^2T^*}{d(r^*)^2}\right] - \frac{3}{r^*}\,{}_0\left(\frac{dT^*}{dr^*}\right) + \\ &\frac{3\beta}{{}_0T^*}\,{}_0\left(\frac{dT^*}{dr^*}\right)^2\Bigg\} - \frac{0.6336(1-k)^2}{k^2}\int_1^{r^*} \frac{dr^*}{{}_0p^*}\frac{d}{dr^*} \\ &\left(\frac{{}_0(\mu^*)^2}{{}_0p^*}\left\{\frac{2}{3}\gamma M_{wa}^2\,{}_0\left(\frac{u^*}{r^*} - \frac{du^*}{dr^*}\right)_0\left(5\frac{u^*}{r^*} + \frac{du^*}{dr^*}\right) + \right.\right. \\ &\left.\left. 2\,{}_0\left[\frac{d^2T^*}{d(r^*)^2}\right] - \frac{1}{r^*}\,{}_0\left(\frac{dT^*}{dr^*}\right) + \frac{2\beta}{{}_0T^*}\,{}_0\left(\frac{dT^*}{dr^*}\right)^2\right\}\right) \end{aligned} \tag{115}$$

is obtained from the second-order terms of equation (85) upon substitution of the zero-order approximation therein.

In terms of ζ, equation (115) becomes

$$\begin{aligned} {}_0\psi = -\frac{0.6336(1-k)^2}{k^2}\Bigg\{&\int_0^{\zeta} \frac{{}_0\mu^*}{{}_0(p^*)^2}\left[\frac{2\gamma}{m} M_{wa}^2\,{}_0\eta\,{}_0\left(\frac{d\omega^*}{d\zeta}\right)_0\left(-\frac{d\omega^*}{d\zeta} - \frac{4m}{\eta}\mu^*\omega^*\right) + \frac{3}{m}\,{}_0\eta (k^2)^{\zeta+\Delta(\zeta)}\,{}_0\left(\frac{d^2T^*}{d\zeta^2}\right) - 6\,{}_0\mu^*(k^2)^{\zeta+\Delta(\zeta)}\,{}_0\left(\frac{dT^*}{d\zeta}\right)\right] d\zeta + \\ &\frac{{}_0\eta}{m}\int_0^{\zeta} \frac{1}{{}_0p^*}\frac{d}{d\zeta}\frac{1}{{}_0p^*}\left[\frac{2}{3}\frac{\gamma}{m}\,{}_0\eta\, M_{wa}^2\,{}_0\left(\frac{d\omega^*}{d\zeta}\right)_0\left(-\frac{d\omega^*}{d\zeta} - 6\frac{m}{\eta}\mu^*\omega^*\right) + 2\frac{{}_0\eta}{m}(k^2)^{\zeta+\Delta(\zeta)}\,{}_0\left(\frac{d^2T^*}{d\zeta^2}\right) - 3\,{}_0\mu^*(k^2)^{\zeta+\Delta(\zeta)}\,{}_0\left(\frac{dT^*}{d\zeta}\right)\right] d\zeta\Bigg\} \end{aligned} \tag{116}$$

Integrating by parts, using equations (C33) and (C34), and neglecting $\Delta(\zeta)$, equation (116) becomes

$$\begin{aligned} {}_0\psi = -\frac{0.6336}{k^2(1+k)^2}\Bigg(&\frac{{}_0\eta}{m}\left[\frac{{}_0\mu^*}{{}_0(p^*)^2}\left(3N_1 - 6N_2k^{2\zeta} - 8N_3\frac{{}_0\eta}{{}_0\mu^*}k^{2\zeta}\right)k^{2\zeta}\right]_0^{\zeta} - \frac{{}_0\eta}{2m}\int_0^{\zeta} {}_0\mu^*\left\{\frac{d}{d\zeta}\left[\frac{1}{{}_0(p^*)^2}\right]\right\}\left(3N_1 - 6N_2k^{2\zeta} - 8N_3\frac{{}_0\eta}{{}_0\mu^*}k^{2\zeta}\right)k^{2\zeta}\,d\zeta + \\ &6\int_0^{\zeta} \frac{{}_0(\mu^*)^2}{{}_0(p^*)^2}\left(N_1 - 2N_2k^{2\zeta} - 2N_4\frac{{}_0\eta}{{}_0\mu^*}k^{2\zeta}\right)k^{2\zeta}\,d\zeta\Bigg) \end{aligned} \tag{117}$$

where

$$\left.\begin{aligned}
N_1&=(1-T_{wb}^*)(1-k^2)^2+(\gamma-1)Pr_{wa}\,M_{wa}^2(1-k^2)-\\
&\quad\frac{8}{3}\gamma k^2M_{wa}^2m\\
N_2&=M_{wa}^2m\left[(\gamma-1)Pr_{wa}-\frac{4}{3}\gamma\right]\\
N_3&=\left[\frac{1}{3}\gamma+(\gamma-1)Pr_{wa}\right]M_{wa}^2m\\
N_4&=\left[\frac{2}{3}\gamma+(\gamma-1)Pr_{wa}\right]M_{wa}^2m
\end{aligned}\right\}\tag{118}$$

As the first approximation put ${}_0\mu^*={}_0\eta$ and assume

$$\frac{1}{{}_0(p^*)^2}=A_0+A_1k^{2\zeta}+A_2k^{4\zeta}\tag{119}$$

with

$$\left.\begin{aligned}
A_0&=1+(2k^2-1)\frac{1-{}_0(p_b^*)^{-2}}{(1-k^2)^2}\\
A_1&=-2k^2\frac{1-{}_0(p_b^*)^{-2}}{(1-k^2)^2}\\
A_2&=\frac{1-{}_0(p_b^*)^{-2}}{(1-k^2)^2}
\end{aligned}\right\}\tag{120}$$

Equations (119) and (120) insure that ${}_0p^*=1$ at $\zeta=0$ and that ${}_0p^*={}_0p_b^*$ and $d_0p^*/d\zeta=0$ at $\zeta=1$. Then from equation (117)

$$\begin{aligned}
{}_0\tilde{\psi}(\zeta)\equiv\frac{{}_0\psi}{{}_0\eta^2}=\frac{0.6336}{k^2(1+k)^2}\Bigg[&\left(\frac{3}{4}A_1A_3+A_0A_4\right)(1-k^{4\zeta})+\\
&(A_2A_3+A_1A_5)(1-k^{6\zeta})+A_2A_6(1-k^{8\zeta})\Bigg]
\end{aligned}\tag{121}$$

where

$$\left.\begin{aligned}
A_3&=(\gamma-1)\,Pr_{wa}M_{wa}^2\frac{1-k^2}{m}+(1-T_{wb}^*)\frac{(1-k^2)^2}{m}-\\
&\quad\frac{8}{3}\gamma k^2M_{wa}^2\\
A_4&=\left[\frac{10}{3}\gamma-8(\gamma-1)Pr_{wa}\right]M_{wa}^2\\
A_5&=\left[\frac{28}{9}\gamma-\frac{23}{3}(\gamma-1)Pr_{wa}\right]M_{wa}^2\\
A_6&=\left[3\gamma-\frac{15}{2}(\gamma-1)Pr_{wa}\right]M_{wa}^2
\end{aligned}\right\}\tag{122}$$

Then for $\zeta=1$

$$\begin{aligned}
{}_0\tilde{\psi}(1)=\frac{0.6336M_{wa}^2}{k^2(1+k)^2}\Bigg\{&\left[\frac{10}{3}\gamma-8(\gamma-1)Pr_{wa}\right](1-k^4)+\\
&\left[1-\frac{1}{{}_0(p_b^*)^2}\right]\Bigg[-\frac{\gamma}{9}(3+26k^2-17k^4)+\\
&\frac{\gamma-1}{6}Pr_{wa}(3+2k^2-47k^4)+\frac{\gamma-1}{2}Pr_{wa}\frac{(1-k^2)(2+k^2)}{m}\Bigg]\Bigg\}+\\
&\frac{0.6336(1-k)^2(2+k^2)}{2k^2m}(1-T_{wb}^*)\left[1-\frac{1}{{}_0(p_b^*)^2}\right]
\end{aligned}\tag{123}$$

The integral in equation (114) remains to be evaluated. It is necessary to assume some relation between μ^* and T^* which, to avoid too much complexity, will be assumed to be the simple relation $\mu^*=T^{*\beta}$ where β is a constant whose value lies between 0.5 and 1.0 (ref. 4, p. 150). Furthermore, in many cases $\beta\approx0.9$ (ref. 14) so the assumption $\beta=1$ or $\mu^*=T^*$ will be made in order to simplify the evaluation of this integral as well as to enable a more direct expansion of all distributions later on.

Equation (114), with the aid of equations (101) and (96), then gives for $\beta=1$

$$\begin{aligned}
{}^2p^*=\left[1+{}_0\tilde{\psi}(\zeta)Kn_e^2\right]\exp\frac{\gamma M_{wa}^2}{2\eta[1-k^2-F(1)]^2}\{&(\omega_a^*-\omega_b^*)^2(1-k^{2\zeta})-\\
&4m(\omega_a^*-\omega_b^*)[k^2\omega_a^*+F(1)\omega_a^*-\omega_b^*]\zeta+\\
&[k^2\omega_a^*+F(1)\omega_a^*-\omega_b^*]^2(k^{-2\zeta}-1)+J(\zeta)\}
\end{aligned}\tag{124}$$

where

$$\begin{aligned}
J(\zeta)=4m(\omega_a^*-\omega_b^*)^2\int_0^\zeta F(\zeta)d\zeta-4m(\omega_a^*-\omega_b^*)[k^2\omega_a^*+&\\
F(1)\omega_a^*-\omega_b^*]\int_0^\zeta k^{-2\zeta}F(\zeta)d\zeta+2m(\omega_a^*-\omega_b^*)^2\times&\\
\int_0^\zeta k^{-2\zeta}[F(\zeta)]^2d\zeta+\sum_{n=1}^{\infty}\frac{(2m)^{n+1}}{n!}\int_0^\zeta\{(\omega_a^*-\omega_b^*)k^\zeta-&\\
[k^2\omega_a^*+F(1)\omega_a^*-\omega_b^*]k^{-\zeta}+(\omega_a^*-\omega_b^*)k^{-\zeta}F(\zeta)\}^2[\Delta(\zeta)]^nd\zeta&
\end{aligned}\tag{125}$$

Upon neglecting all terms containing $\Delta(\zeta)$, the first approximation to $J(\zeta)$ is

$$J(\zeta)={}^1J(\zeta)=0\tag{126}$$

while, neglecting terms containing the second and higher powers of $\Delta(\zeta)$, the second approximation becomes

$$\begin{aligned}
{}^2J(\zeta)=(\omega_a^*-\omega_b^*)^2J_1(\zeta)+\left(\frac{k^2\omega_a^*-\omega_b^*}{k^2}\right)^2J_2(\zeta)-&\\
2(\omega_a^*-\omega_b^*)\left(\omega_a^*-\frac{\omega_b^*}{k^2}\right)k^{2-2\zeta^2}F(\zeta)&
\end{aligned}\tag{127}$$

where

$$\left.\begin{aligned} J_1(\zeta)&={}^2F(\zeta)+4m\int_0^\zeta {}^2F(\zeta)d\zeta \\ J_2(\zeta)&=4k^4m^2\int_0^\zeta k^{-2\zeta}\,{}^2\Delta(\zeta)d\zeta \end{aligned}\right\} \tag{128}$$

Equations (124) for the pressure distribution, (101) for the velocity distribution, and (103) for the temperature distribution in terms of the independent variable ζ represent the solution to the problem, valid to terms in the square of the Knudsen number, provided the assumptions of constant specific heat, of constant Prandtl number and coefficient of viscosity, as well as of heat conductivity, proportional to the first power of the absolute temperature, are valid.

The solution (eq. (124)) for the pressure is more simple than Schamberg's solution in the sense that the present assumption $\mu^*=T^*$ eliminates the explicit dependence of the integral in equation (114) upon T^*. Figure 2 shows the dependence of ${}_0\tilde{\psi}(1)$ upon k as given by equation (123) for the values of the physical parameters selected for air in the section "Case of Air" and discussed there.

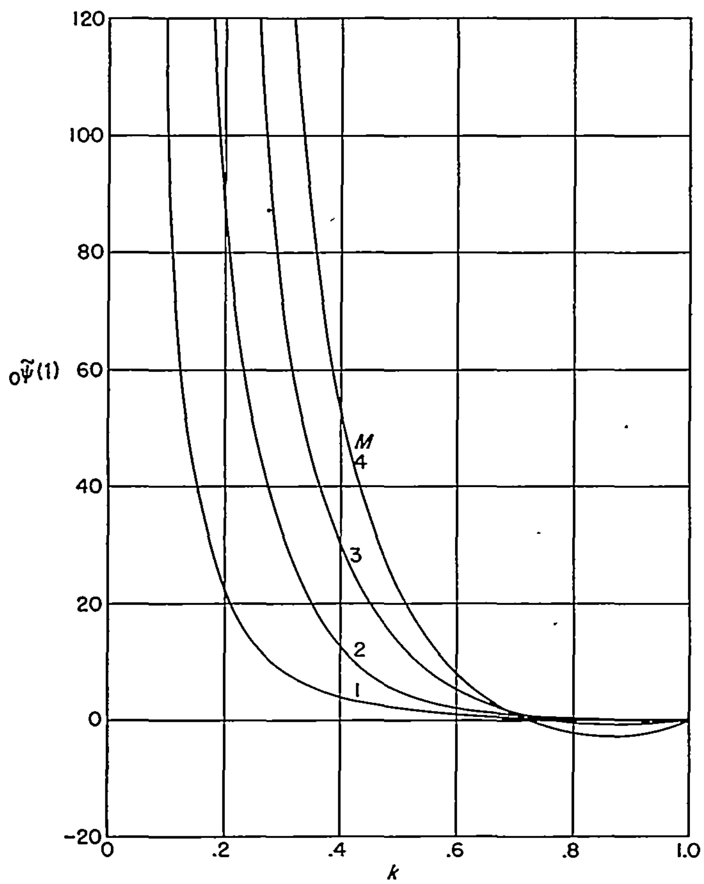


FIGURE 2.—Burnett terms correction to pressure ratio p_b^* against diameter ratio k for various values of Mach number M_{wa}. $Pr=0.715$; $\gamma=1.400$; $\beta=1$; $T_{wb}^*=1$; $\alpha=0.900$; $\sigma=1$.

The explicit determination of the expansions of the coefficient η and of the variables ξ, $\Delta(\zeta)$, $F(\zeta)$, $G(\zeta)$, and so forth in powers of Knudsen number is carried out in appendix D.

FRICTION COEFFICIENT, SLIP VELOCITY, HEAT TRANSFER, TEMPERATURE JUMP, AND PRESSURE RATIO

FRICTION COEFFICIENT

It is convenient to define the friction coefficient C_f as

$$C_f=\frac{\tau_{r\phi}}{\frac{1}{2}\rho_a U^2} \tag{129}$$

The expression for $\tau_{r\phi}$ is obtained from equation (50) which, upon using equations (81), becomes

$$\tau_{r\phi}=-\left(\frac{\mu_{wa}U}{a}\right)r^*\mu^*\frac{d}{dr^*}\left(\frac{u^*}{r^*}\right) \tag{130}$$

From equations (89) and (90)

$$r^*\mu^*\frac{d}{dr^*}=\frac{\eta}{m}\frac{d}{d\zeta}$$

Therefore

$$\tau_{r\phi}=-\frac{\mu_e U}{am}\frac{d\omega^*}{d\zeta} \tag{131}$$

where

$$\mu_e=\eta\mu_{wa} \tag{132}$$

Substituting equation (131) into equation (129) and using equations (79), (80), and (81),

$$Re_eC_f=-\frac{2(1-k)}{km}\frac{d\omega^*}{d\zeta} \tag{133}$$

where

$$\eta Re_e=Re_{wa} \tag{134}$$

From equations (C21), (C27), and (C29) the value at the wall of the inner cylinder becomes

$$Re_eC_{fa}=\frac{4[1+({}_1\omega_a^*-{}_1\omega_b^*)Kn_e+({}_2\omega_a^*-{}_2\omega_b^*)Kn_e^2]}{k\{1+k-[F(1)/(1-k)]\}} \tag{135}$$

Similarly, at the wall of the outer cylinder

$$Re_eC_{fb}=\frac{4k[1+({}_1\omega_a^*-{}_1\omega_b^*)Kn_e+({}_2\omega_a^*-{}_2\omega_b^*)Kn_e^2]}{1+k-[F(1)/(1-k)]} \tag{136}$$

Since there must be equal torque on both the inner and the outer cylinders, $C_{fb}/C_{fa}=k^2$, as shown by equations (135) and (136). If

$$C_f={}_0C_f(1+{}_1C_fKn_e+{}_2C_fKn_e^2) \tag{137}$$

then

$$Re_e\,{}_0C_{fa}=\frac{4}{k\{1+k-[{}_0F(1)/(1-k)]\}} \tag{138}$$

$$Re_{e0}C_{f0}=\frac{4k}{1+k-[_0F(1)/(1-k)]} \tag{139}$$

SLIP VELOCITY

From equations (101), (81), (C28), and (C29), the slip velocity at the inner wall is

$$\frac{u_a-U}{U}={}_1u_a{}^*Kn_e+{}_2u_a{}^*Kn_e{}^2={}_1\omega_a{}^*Kn_e+{}_2\omega_a{}^*Kn_e{}^2 \tag{140}$$

Similarly, at the outer wall it is

$$\frac{u_b}{U}={}_1u_b{}^*Kn_e+{}_2u_b{}^*Kn_e{}^2=({}_1\omega_b{}^*/k)Kn_e+({}_2\omega_b{}^*/k)Kn_e{}^2 \tag{141}$$

where ${}_1\omega_a{}^*$, ${}_1\omega_b{}^*$, ${}_2\omega_a{}^*$, and ${}_2\omega_b{}^*$ are given by equations (C30) and (C31).

HEAT TRANSFER

It is of interest to calculate the heat transfer between the rarefied gas and the cylinders. If the dimensionless heat transfer q^* is defined by

$$q^*=\frac{hq_r}{\lambda_eT_{wa}} \tag{142}$$

where

$$\lambda_e=\eta\lambda_{wa} \tag{143}$$

then from equations (53) and (81),

$$-q^*=\frac{(1-k)\lambda^*}{k\eta}\frac{dT^*}{dr^*} \tag{144}$$

For Pr/c_p=Constant, $\lambda^*=\mu^*$; so, using equations (89) and (90),

$$q^*=-\frac{1-k}{kmr^*}\frac{dT^*}{d\zeta} \tag{145}$$

From equation (145) and using equations (C22), (C27), and (C29), the dimensionless heat transfer at the inner cylinder is

$$q_a{}^*=\frac{1-k}{km}\Big\{1-T_{wb}{}^*+({}_1T_a{}^*-T_{wb}{}^*{}_1T_b{}^*)Kn_e+({}_2T_a{}^*-T_{wb}{}^*{}_2T_b{}^*)Kn_e{}^2+(\gamma-1)PrM_{wa}{}^2\frac{1-k^2-2m-G'(0)}{[1-k^2-F(1)]^2}[1+2({}_1\omega_a{}^*-{}_1\omega_b{}^*)Kn_e+ ({}_1\omega_a{}^*-{}_1\omega_b{}^*)^2Kn_e{}^2+2({}_2\omega_a{}^*-{}_2\omega_b{}^*)Kn_e{}^2]\Big\} \tag{146}$$

Similarly, that at the outer cylinder is

$$q_b{}^*=\frac{1-k}{m}\Big\{1-T_{wb}{}^*+({}_1T_a{}^*-T_{wb}{}^*{}_1T_b{}^*)Kn_e+({}_2T_a{}^*-T_{wb}{}^*{}_2T_b{}^*)Kn_e{}^2+(\gamma-1)PrM_{wa}{}^2\frac{1-k^2-2k^2m-G'(1)}{[1-k^2-F(1)]^2}[1+2({}_1\omega_a{}^*-{}_1\omega_b{}^*)Kn_e+ ({}_1\omega_a{}^*-{}_1\omega_b{}^*)^2Kn_e{}^2+2({}_2\omega_a{}^*-{}_2\omega_b{}^*)Kn_e{}^2]\Big\} \tag{147}$$

If

$$q^*={}_0q^*(1+{}_1q^*Kn_e+{}_2q^*Kn_e{}^2) \tag{148}$$

then

$${}_0q_a{}^*=\frac{1-k}{km}\Big\{1-T_{wb}{}^*+(\gamma-1)Pr_{wa}M_{wa}{}^2\frac{1-k^2-2m-{}_0G'(0)}{[1-k^2-{}_0F(1)]^2}\Big\} \tag{149}$$

$${}_0q_b{}^*=\frac{1-k}{m}\Big\{1-T_{wb}{}^*+(\gamma-1)Pr_{wa}M_{wa}{}^2\frac{1-k^2-2k^2m-{}_0G'(1)}{[1-k^2-{}_0F(1)]^2}\Big\} \tag{150}$$

The expressions for ${}_1T_a{}^*$, ${}_1T_b{}^*$, ${}_2T_a{}^*$, and ${}_2T_b{}^*$ are given by equations (C30) and (C31).

TEMPERATURE JUMP

From equation (C27), using equations (C29) and (81), the temperature jump at the inner cylinder is

$$\frac{T_a-T_{wa}}{T_{wa}}={}_1T_a{}^*Kn_e+{}_2T_a{}^*Kn_e{}^2 \tag{151}$$

Similarly, the temperature jump at the outer cylinder is

$$\frac{T_b-T_{wb}}{T_{wb}}={}_1T_b{}^*Kn_e+{}_2T_b{}^*Kn_e{}^2 \tag{152}$$

PRESSURE RATIO

The ratio of the hydrostatic pressure at the wall of the inner cylinder to that at the outer cylinder is obtained from equation (114), upon using equations (121) and (81), as follows:

$$\frac{p_b}{p_a}=[1+Kn_e{}^2{}_0\tilde{\psi}(1)]\exp\frac{\gamma M_{wa}{}^2m}{\eta}\int_0^1\frac{\mu^*(\omega^*)^2d\zeta}{T^*(k^2)^{\zeta+\Delta\zeta}} \tag{153}$$

For the case $\mu\approx T$, equations (124) and (81) give equation (153) the form

$$\frac{p_b}{p_a}=[1+Kn_e{}^2{}_0\tilde{\psi}(1)]\exp\frac{\gamma M_{wa}{}^2}{2\eta[1-k^2-F(1)]^2}\{[(1-k^4-4k^2m)+ 2(1-k^2-2m)F(1)+(k^{-2}-1)F^2(1)](\omega_a{}^*)^2+ [-2(1-k^2)+2(1+k^2)m+(1-k^{-2}+2m)F(1)]2\omega_a{}^*\omega_b{}^*+ (k^{-2}-k^2-4m)(\omega_b{}^*)^2+J(1)\} \tag{154}$$

In general $J(1)$ is a small number and can be replaced by

${}_0^2J(1)$, which is given by equation (D41). From equations (154) and (C29)

$$ {}_0^1\left(\frac{p_b}{p_a}\right)=\exp\frac{\gamma M_{wa}^2}{2\,{}_0\eta\,[1-k^2-{}_0F(1)]^2}[(1-k^4-4k^2m)+2(1-k^2-2m)\,{}_0F(1)+(k^{-2}-1)\,{}_0F^2(1)+{}_0^2J(1)] \quad (155) $$

For the case of equal wall temperature, $T_{wb}^*=1$ and, from equations (D23) and (C29), ${}_0F(1)=0$. Therefore

$$ {}_0^1\left(\frac{p_b}{p_a}\right)=\exp\frac{\gamma M_{wa}^2}{2\,{}_0\eta(1-k^2)^2}[(1-k^4-4k^2m)+{}_0^2J(1)] \quad (156) $$

Or in terms of ϵ

$$ {}_0^1\left(\frac{p_b}{p_a}\right)=\exp\frac{\gamma M_{wa}^2\epsilon}{6\,{}_0\eta}\left[\left(1+\frac{1}{2}\epsilon+\frac{3}{10}\epsilon^2+\frac{1}{5}\epsilon^3+\frac{1}{7}\epsilon^4+\ .\ .\ .\ \right)-\frac{(\gamma-1)Pr\,M_{wa}^2\epsilon}{120\,{}_0^1\eta}\left(1+\frac{3}{2}\epsilon+\frac{143}{84}\epsilon^2+\ .\ .\ .\ \right)\right] \quad (156a) $$

From equations (C28) and (154), using equations (81), (C29), and (C27) and neglecting the first-order corrections to ${}_0F(1)$ and ${}_0J(1)$, it follows that

$$ {}_1p_b^*=\frac{\gamma M_{wa}^2}{{}_0\eta\,[1-k^2-{}_0F(1)]^2}\Big\{[(1-k^4-4k^2m)+2(1-k^2-2m)\,{}_0F(1)+(k^{-2}-1)\,{}_0F^2(1)]\left({}_1\omega_a^*-\frac{1}{2}{}_1\eta^*\right)+[-2(1-k^2)+2(1+k^2)m-(k^{-2}-1-2m)\,{}_0F(1)]\,{}_1\omega_b^*-\frac{1}{2}{}_1\eta^*\,{}_0^2J(1)\Big\} \quad (157) $$

For the case of equal wall temperature ${}_0F(1)=0$ and

$$ {}_1p_b^*=\frac{\gamma M_{wa}^2}{{}_0\eta\,(1-k^2)^2}\Big\{(1-k^4-4k^2m)\left({}_1\omega_a^*-\frac{1}{2}{}_1\eta^*\right)+2[-1+k^2+(1+k^2)m]\,{}_1\omega_b^*-\frac{1}{2}{}_1\eta^*\,{}_0^2J(1)\Big\} \quad (158) $$

$$ {}_1p_b^*=\frac{\gamma M_{wa}^2\epsilon}{3\,{}_0\eta}\left[\left(1+\frac{1}{2}\epsilon+\frac{3}{10}\epsilon^2+\frac{1}{5}\epsilon^3+\frac{1}{7}\epsilon^4+\ .\ .\ .\right)\left({}_1\omega_a^*-\frac{1}{2}{}_1\eta^*\right)+\frac{1}{2}\left(1+\epsilon+\frac{9}{10}\epsilon^2+\frac{4}{5}\epsilon^3+\frac{5}{7}\epsilon^4+\ .\ .\ .\right){}_1\omega_b^*+\frac{\epsilon}{80}\frac{{}_1\eta^*(\gamma-1)PrM_{wa}^2}{{}_0^1\eta}\left(\frac{1}{3}+\frac{1}{2}\epsilon+\frac{143}{252}\epsilon^2+\ .\ .\ .\right)\right] \quad (158a) $$

The use of effective values μ_e and λ_e and hence of an effective Reynolds number Re_e, similar to the effective Knudsen number Kn_e defined by equation (109), is a natural consequence of the form of the expressions for the skin-friction coefficient, the velocity of slip, the heat transfer, and the temperature jump. The use of these effective values gives expressions closest in form to those of Schamberg and agrees with his when the functions F and G' are zero. Since η as determined in appendix D will, for equal wall temperatures, increase with the Mach number M_{wa} and "curvature" $1/k$ as shown in figure 3, so also will μ_e and λ_e.

The limiting values of all expressions when $k=1$ or $\epsilon\rightarrow 0$ agree exactly with the plane Couette flow solution of reference 7.

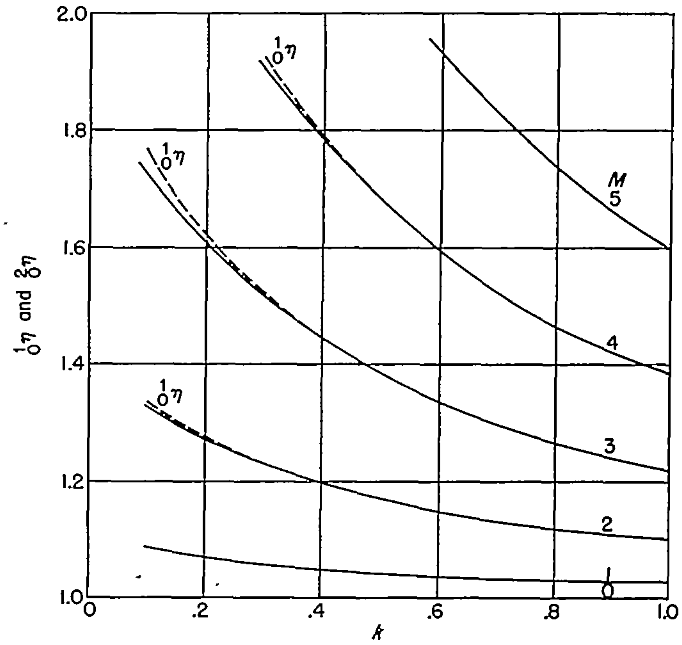


FIGURE 3.—Variations of ${}_0^1\eta$ and ${}_0^2\eta$ with diameter ratio k for various values of Mach number M_{wa}. $Pr=0.715$; $\alpha=1.400$; $\beta=1$; $T_{wb}^*=1$.

CASE OF AIR

Assuming that the gas is air and that the cylinders are made of metal such as aluminum or brass, the required physical constants are given as follows:

For the ratio of the specific heats take

$$ \gamma=1.400 \quad (159) $$

a value which is reasonably accurate and has the advantage of computational simplicity. It also agrees with the value given by the kinetic theory for a diatomic gas whose molecules have five degrees of freedom.

The kinetic theory gives, for the Prandtl number, $Pr=4\gamma/(9\gamma-5)=0.737$ at 0° C for a diatomic gas with $\gamma=1.400$ (ref. 4, p. 182). For air, various values have been used in the study of the laminar boundary layer in compressible flow by different investigators. The value $Pr=0.725$ was used by Crocco and Conforto (refs. 15 and 16) in 1941; $Pr=0.733$, by Brainerd and Emmons (refs. 17 and 18) in 1942; $Pr=0.750$, by Schamberg (ref. 3) in 1947; and $Pr=0.715$, by Cope and Hartree (ref. 14) in 1948. The value $Pr=1$ was also used by several writers (refs. 19 to 21) for mathematical simplicity. In the calculations to follow the value used will be

$$ Pr=0.715 \quad (160) $$

as suggested by Cope and Hartree in 1948, based on the latest data for air given by Kaye and Laby (ref. 22), 1941.

Maxwell's reflection coefficient σ is given for air on machined brass by R. A. Millikan (see ref. 4, p. 299) as

$$ \sigma=1.00 \quad (161) $$

The accommodation coefficient α has, according to M. Wiedmann (see ref. 1, p. 658), the average value of

$$ \alpha=0.900 \quad (162) $$

for air on metal. This value is also used by Schamberg (ref. 3).

With these physical constants, equations (64) give the following constants for slip flow:

$$\left.\begin{aligned}a_1&=1.253\\c_1'&=2.498\\e_1&=0.251\\e_6&=1.911\\e_8&=-5.491\end{aligned}\right\}\tag{163}$$

It is noted that, since air is not a monatomic gas, c_1' is used instead of c_1.

For the case that $\mu\approx T$,

$$\beta=1\tag{164}$$

Upon using equations (C29), (159), and (160), equation (D16) gives

$${}^1_0\eta=\frac{1}{2}(1+T_{wb}^*)+\frac{0.143}{1-k^2}\left(\frac{1+k^2}{1-k^2}-\frac{1}{m}\right)M_{wa}^2\tag{165}$$

For the case of equal wall temperatures,

$$\left.\begin{aligned}T_{wb}^*&=1\\{}_0F(1)&=0\end{aligned}\right\}\tag{166}$$

Similarly, using equations (C29), (159), (160), (164) to (166), and (D28), equation (D9) gives

$${}^2_0\eta={}^1_0\eta-\frac{M_{wa}^4}{{}^1_0\eta}\frac{0.02045}{(1-k^2)^2}\left[\frac{1-8k^2+k^4}{2(1-k^2)^2}-\frac{9(1+k^2)}{4(1-k^2)m}+\frac{3}{m^2}\right]\tag{167}$$

The variations of both ${}^1_0\eta$ and ${}^2_0\eta$ for various values of the ratio k of the radii of the cylinders at various Mach numbers and equal wall temperatures with the ratio of the specific heats $\gamma=1.400$, Prandtl number $Pr=0.715$, and the viscosity index $\beta=1$ are given in figure 3.

It is seen that, for $k>0.40$ and $M<4$, ${}^1_0\eta$ and ${}^2_0\eta$ have practically the same value, and therefore equation (165) can be used for calculating ${}^2_0\eta$ instead of equation (167).

At smaller values of k and higher Mach number ${}^2_0\eta<{}^1_0\eta$; that is, the correction tends to decrease the value of η.

From equation (156), upon using equations (D41), (159), (160), (164), and (166),

$$\log_e {}^1_0p_b^*=\frac{M_{wa}^2}{{}_0\eta}\frac{0.7000}{(1-k^2)^2}(1-k^4-4k^2m)-\frac{M_{wa}^4}{{}_0\eta^2}\frac{0.2002}{(1-k^2)^2}\left[\frac{1}{2}(1+k^2)+\frac{2k^2m}{1-k^2}-\frac{1-k^2}{m}\right]\tag{168}$$

The variations of ${}^1_0p_b^*$ with the ratio k at various Mach numbers and at equal wall temperatures with $\gamma=1.400$, $Pr=0.715$, and $\beta=1$ are given in figure 4. It is seen that the pressure ratio ${}^1_0p_b^*$ increases rapidly with the curvature $1/k$, especially at high Mach numbers.

Upon using equations (C30), (166), and (163), equation (140) gives

$${}_1u_a^*=-\frac{1.995}{k(1+k)}\tag{169}$$

Similarly, equation (141) gives

$${}_1u_b^*=\frac{1.995k}{(1+k)\,{}_0p_b^*}\tag{170}$$

The variations of ${}_1u_a^*$ and ${}_1u_b^*$ with the ratio k are given in figures 5 and 6, respectively. It is noted that ${}_1u_a^*$ is independent of the Mach number M_{wa}, while ${}_1u_b^*$ depends on M_{wa} through ${}_0p_b^*$. As k decreases or the curvature increases, it is seen that u_a^* becomes more negative or the slip velocity at the inner cylinder increases in magnitude while the slip velocity at the outer cylinder decreases.

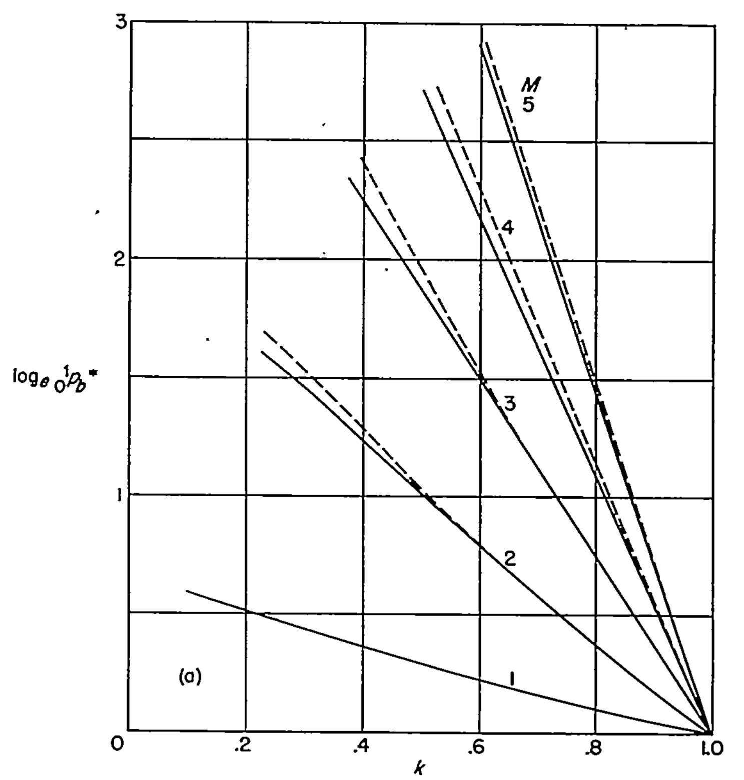
Equations (C29), (166), (159), (160), (163), (D31), and (C30) give

$${}_1T_a^*=\frac{0.569}{k(1+k)}\left(\frac{2}{1-k^2}-\frac{1}{m}\right)M_{wa}^2+\frac{0.163}{k(1+k)(1-k^2)}\left[\frac{1+k^2}{4(1-k^2)m}+\frac{k^2}{(1-k^2)^2}-\frac{1}{2m^2}\right]\frac{M_{wa}^4}{{}^1_0\eta}\tag{171}$$

and

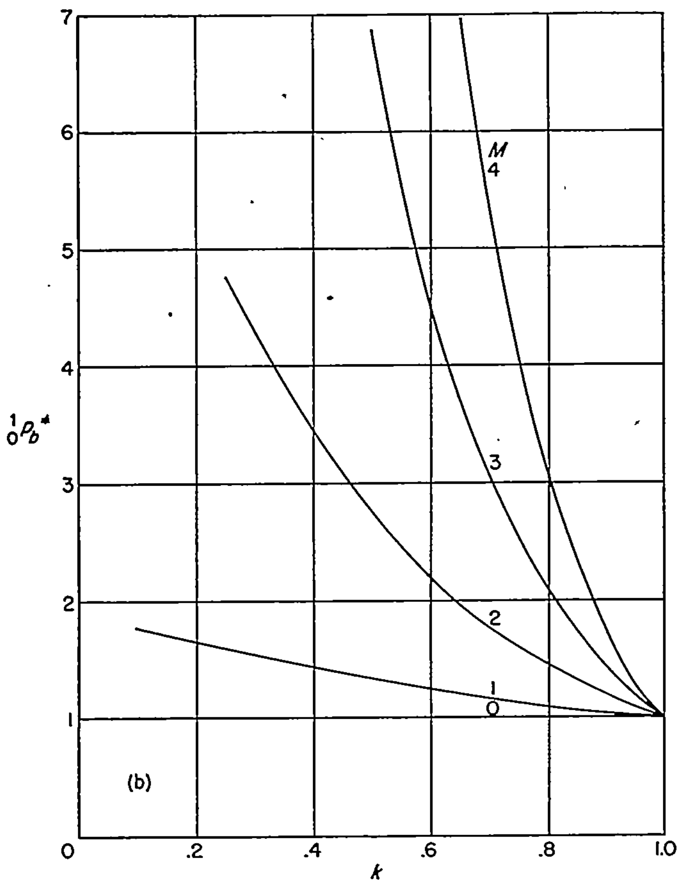
$${}_1T_b^*=\frac{0.569}{1+k}\left(\frac{1}{m}-\frac{2k^2}{1-k^2}\right)\frac{M_{wa}^2}{{}_0p_b^*}-\frac{0.163}{(1+k)(1-k^2)}\left[\frac{1+k^2}{4(1-k^2)m}+\frac{k^2}{(1-k^2)^2}-\frac{1}{2m^2}\right]\frac{M_{wa}^4}{{}^1_0\eta\,{}_0p_b^*}\tag{172}$$

The variations of ${}_1T_a^*$ and ${}_1T_b^*$ with the ratio k for various Mach numbers and equal wall temperatures with $\gamma=1.400$, $Pr=0.175$, and $\beta=1$ are given in figures 7 and 8, respectively. It is seen that as k decreases ${}_1T_a^*$ becomes much more important than ${}_1T_b^*$, the latter going to zero with k for all values of M.



(a) $\log_e {}^1_0p_b^*$ against k. Dashed curves neglect M^4 terms.

FIGURE 4.—Variations of zero-order pressure ratio with diameter ratio k for various values of Mach number. $Pr=0.715$; $\gamma=1.400$; $\beta=1$; $T_{wb}^*=1$.



(b) $^{1}_{0}p_b{}^*$ against k.

FIGURE 4.—Concluded.

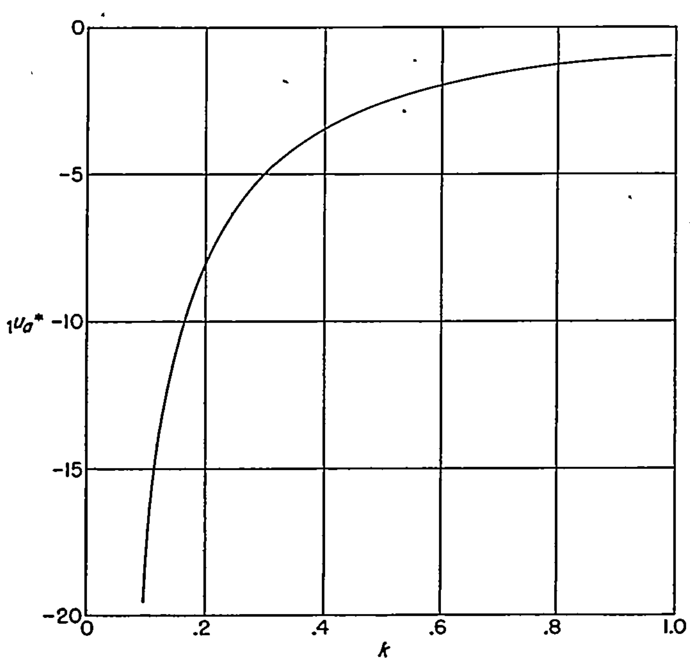


FIGURE 5.—First-order slip velocity at inner cylinder against diameter ratio k. $Pr=0.715$; $\gamma=1.400$; $\beta=1$; $T_{wb}{}^*=1$; $\sigma=1$.

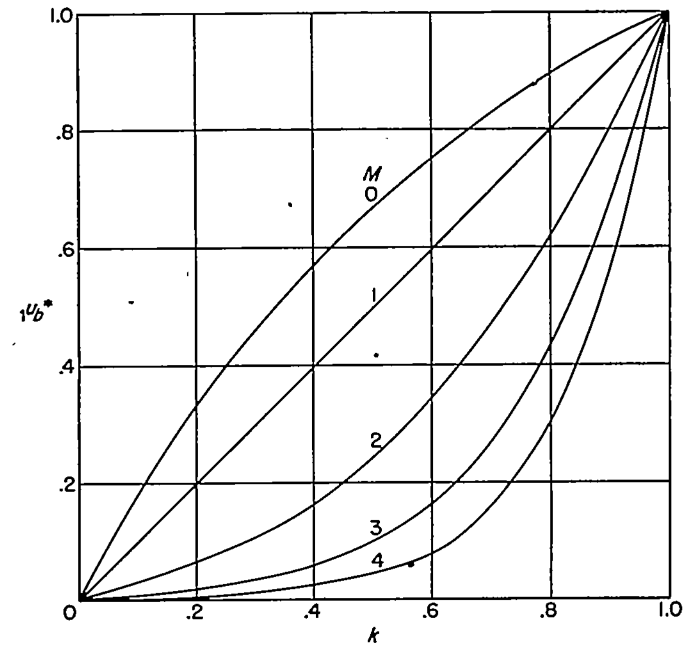


FIGURE 6.—First-order slip velocity at outer cylinder against diameter ratio k. $Pr=0.715$; $\gamma=1.400$; $\beta=1$; $T_{wb}{}^*=1$; $\sigma=1$.

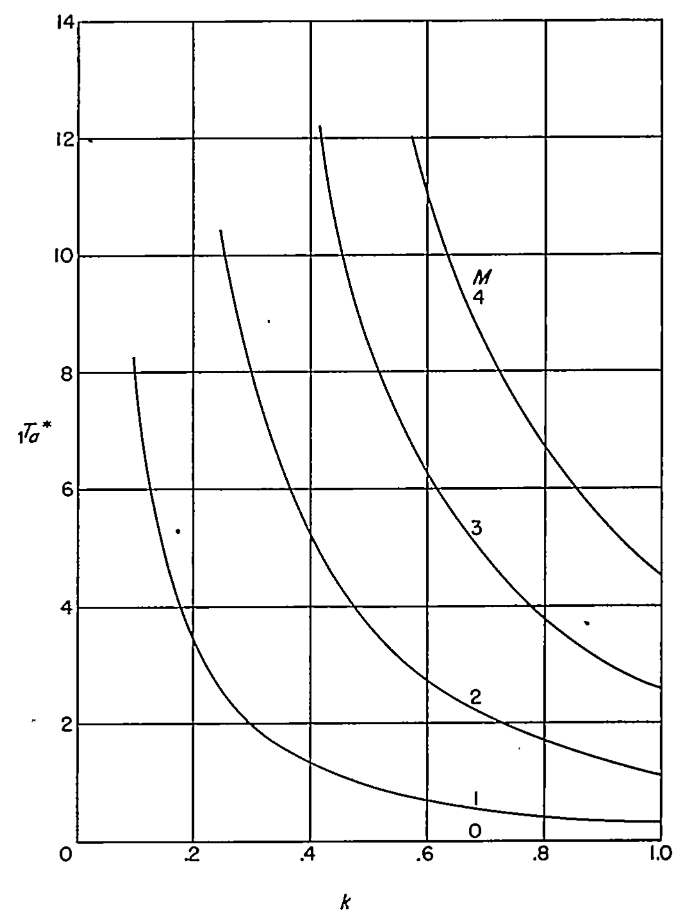


FIGURE 7.—First-order temperature jump at inner cylinder against diameter ratio k. $Pr=0.715$; $\gamma=1.400$; $\beta=1$; $T_{wb}{}^*=1$; $\alpha=0.900$.

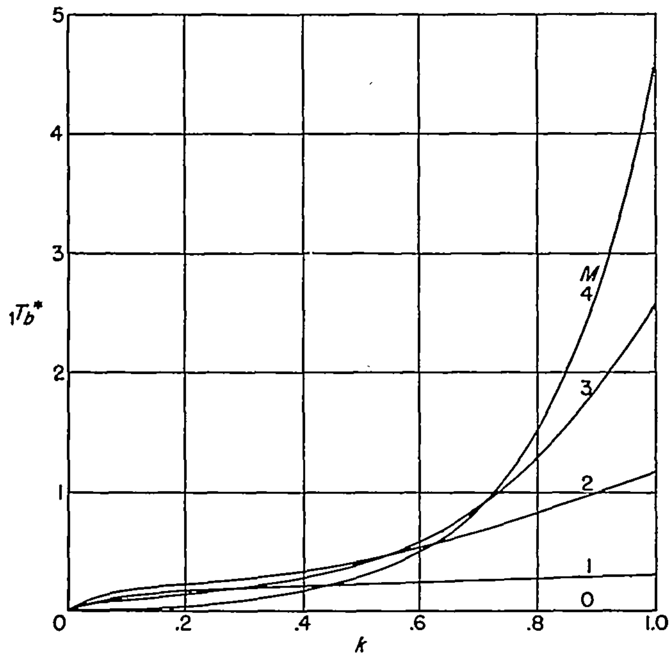


FIGURE 8.—First-order temperature jump at outer cylinder against diameter ratio k. $Pr=0.715$; $\gamma=1.400$; $\beta=1$; $T_{wb}^*=1$; $\alpha=0.900$.

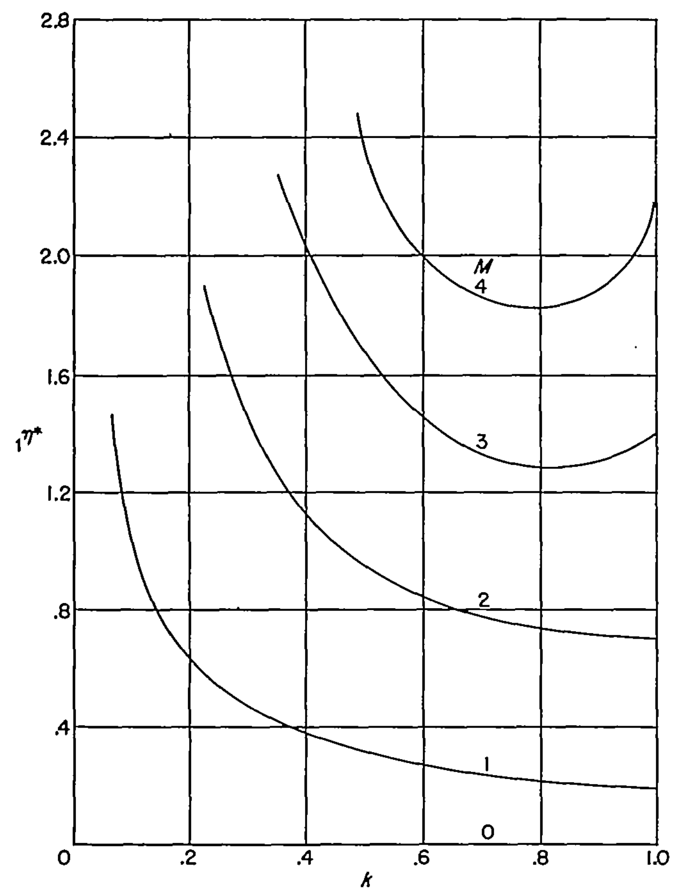


FIGURE 9.—First-order correction to parameter η against diameter ratio k for various values of Mach number M_{wa}. $Pr=0.715$; $\gamma=1.400$; $\beta=1$; $T_{wb}^*=1$; $\alpha=0.900$; $\sigma=1$.

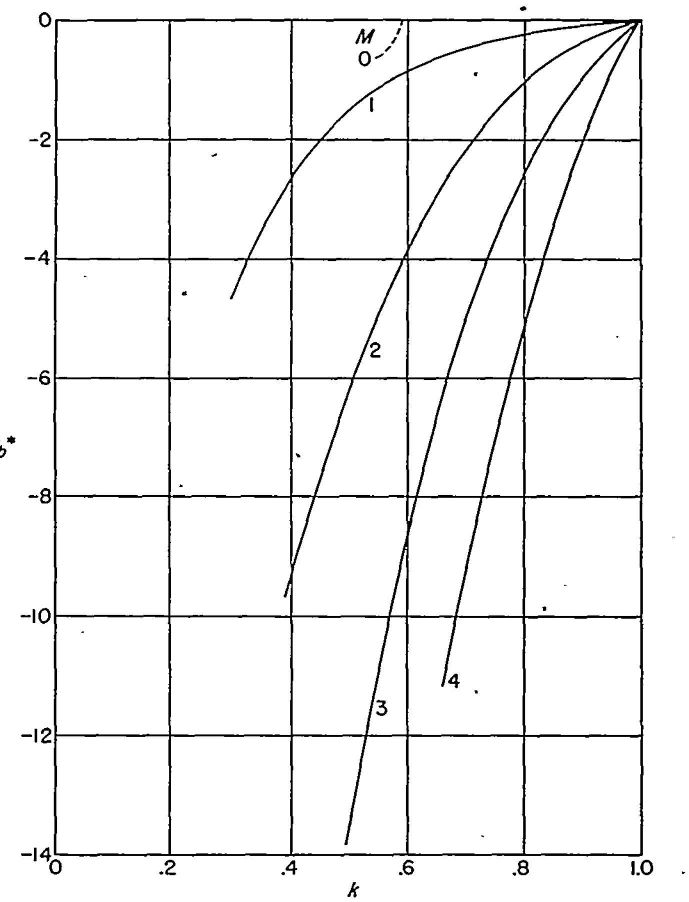


FIGURE 10.—First-order correction to pressure ratio $_1p_b^*$ against diameter ratio k. $Pr=0.715$; $\gamma=1.400$; $\beta=1$; $T_{wb}^*=1$; $\alpha=0.900$; $\sigma=1$.

In figure 8 the anomalous behavior of the curves for small and large values of k is due to the large exponential values of $_0p_b^*$ for increasing Mach number at all values of k except in the neighborhood of $k=1$. In fact, for $k=1$, $_1T_b^*=0.2845M_{wa}^2$, independent of $_0p_b^*$. This is seen upon expanding equation (172) in powers of $\epsilon=1-k^2$ and letting $\epsilon\rightarrow 0$.

From equation (D10), the variations of $_1\eta^*$ with the ratio k for various Mach numbers and equal wall temperatures are found and plotted in figure 9. Since $_1\eta^*>0$, it follows from equations (C28) that η increases with the Knudsen number.

Making use of equations (D41), (159), (160), and (166), equation (158) gives

$$\begin{aligned}{}_1p_b^*=&0.700\,\frac{1-k^4-4k^2m}{(1-k^2)^2}\,\frac{M_{wa}^2}{_0\eta}\,(2\,{}_1\omega_a^*-{}_1\eta^*)+\\&2.800\,\frac{-1+k^2+(1+k^2)m}{(1-k^2)^2}\,\frac{M_{wa}^2}{_0\eta}\,{}_1\omega_b^*+\\&0.2002\,\frac{1}{(1-k^2)^2}\left[\frac{1}{2}(1+k^2)+\frac{2k^2m}{1-k^2}-\frac{1-k^2}{m}\right]\frac{M_{wa}^4}{_0\eta^2}\,{}_1\eta^*\end{aligned}\tag{173}$$

The variations of $_1p_b^*$ with the ratio k for various Mach numbers are given in figure 10. Since $_1p_b^*<0$, it follows that the pressure ratio p_b^* decreases as the rarefaction of the gas, that is, the Knudsen number, increases.

From equations (135) to (139) and (146) to (150), the variations of $Re_{e0}C_{fa}$ and $Re_{e0}C_{fb}$, ${}_1C_{fa}$ and ${}_1C_{fb}$, ${}_0q_a^*$, and ${}_0q_b^*$ with the ratio k are found and plotted in figures 11, 12, 13, and 14, respectively.

The zero-order values of both the skin friction and the heat transfer are observed to increase in absolute magnitude at the inner and decrease at the outer cylinder with increasing curvature.

Upon using equations (159), (160), and (166), equation (123) yields

$$
{}_0\tilde{\psi}(1)=\frac{1.51\,M_{wa}^2}{k^2(1+k)^2}\{(1-k^4)+[1-{}_0(p_b^*)^{-2}][-0.136-1.660k^2+0.169k^4+0.060(1-k^2)(2+k^2)/m]\} \quad (174)
$$

The variations of ${}_0\tilde{\psi}(1)$ with the ratio k for various Mach numbers and equal wall temperatures are given in figure 2. It is seen that ${}_0\tilde{\psi}(1)>0$, for sufficiently small k, and it increases rapidly as k decreases. It follows that the effect of the Burnett terms in the differential equation tends to increase the pressure ratio p_b^*.

The coefficient of the last term in equation (123) is plotted against k in figure 15. For the case of equal wall temperature, this term becomes zero.

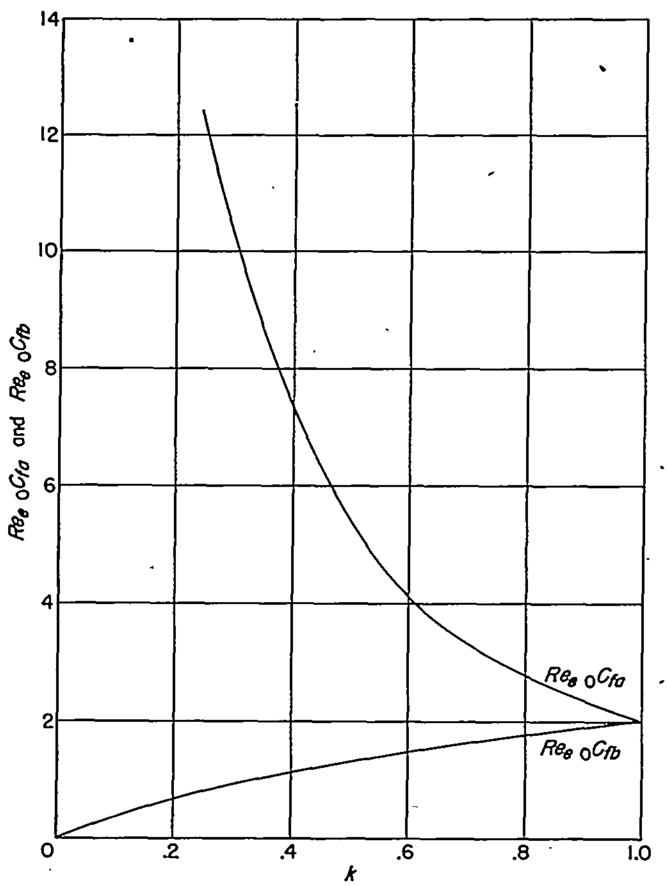


FIGURE 11.—Zero-order friction coefficient at inner and outer cylinders against diameter ratio k. $Pr=0.715$; $\gamma=1.400$; $\beta=1$; $T_{wb}^*=1$; $\alpha=0.900$; $\sigma=1$.

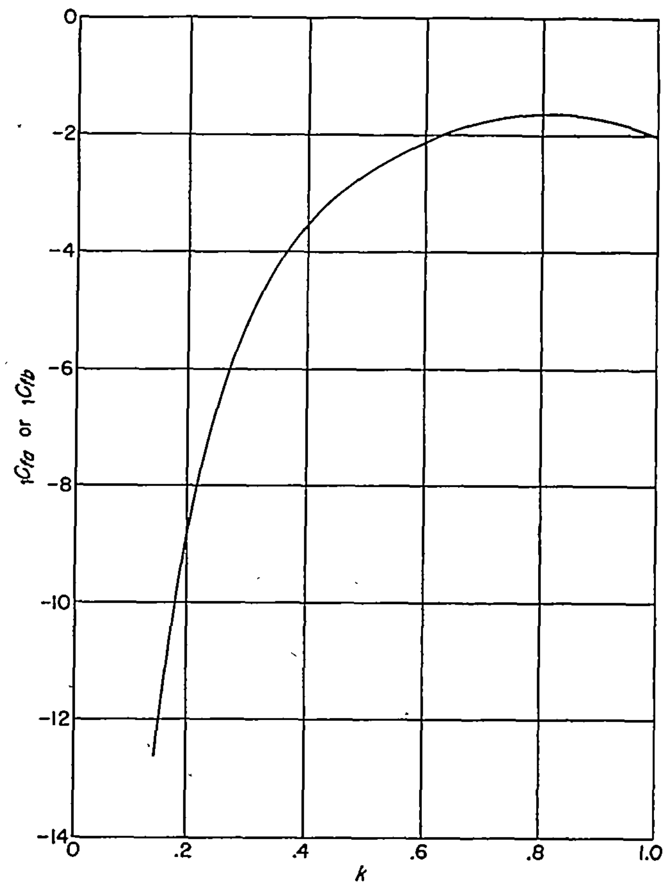


FIGURE 12.—First-order correction to friction coefficient for both inner and outer cylinders for various values of diameter ratio k. $Pr=0.715$; $\gamma=1.400$; $\beta=1$; $T_{wb}^*=1$; $\alpha=0.900$; $\sigma=1$.

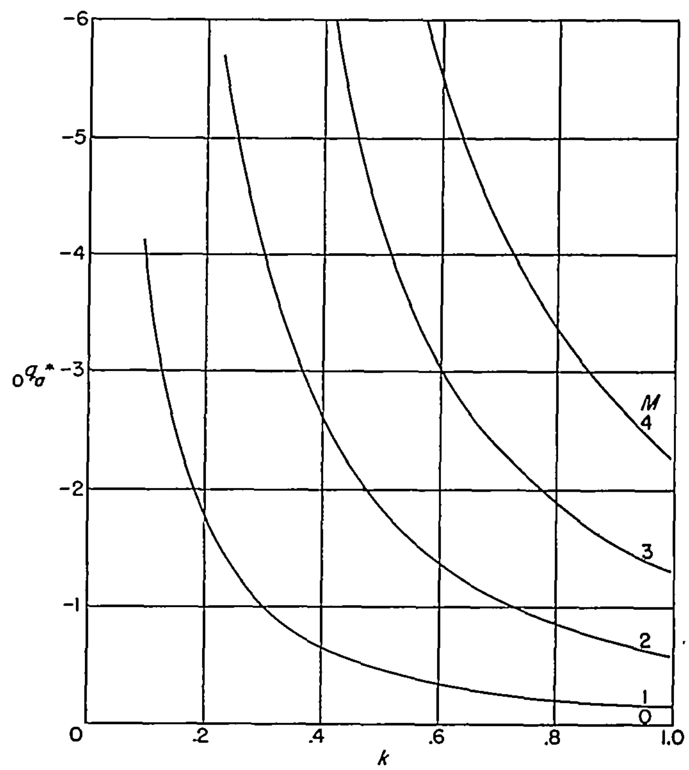


FIGURE 13.—Zero-order heat transfer at inner cylinder against diameter ratio k. $Pr=0.715$; $\gamma=1.400$; $\beta=1$; $T_{wb}^*=1$; $\alpha=0.900$; $\sigma=1$.

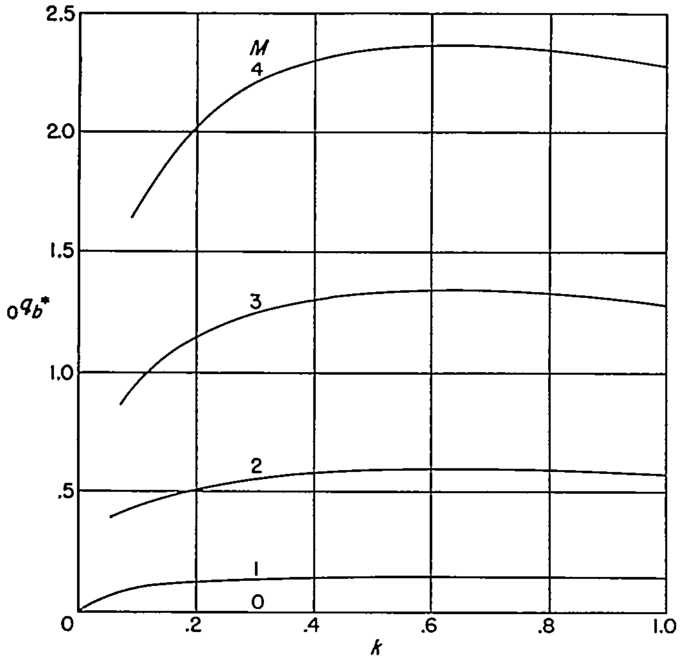


FIGURE 14.—Zero-order heat transfer at outer cylinder against diameter ratio k. $Pr=0.715$; $\gamma=1.400$; $\beta=1$; $T_{wb}^*=1$; $\alpha=0.900$; $\sigma=1$.

Consider in more detail the special case

$$k=0.5 \tag{175}$$

and

$$M_{wa}=2.0 \tag{176}$$

$$\left.\begin{array}{ll}
\text{From figure 3} & {}_0\eta=1.17 \\
\text{From figure 4} & {}_0^1p_b^*=2.77 \\
\text{From figure 5} & {}_1u_a^*=-2.67 \\
\text{hence} & {}_1\omega_a^*=-2.67 \\
\text{From figure 6} & {}_1u_b^*=0.24 \\
\text{hence} & {}_1\omega_b^*=0.12 \\
\text{From figure 7} & {}_1T_a^*=3.73 \\
\text{From figure 8} & {}_1T_b^*=0.42 \\
\text{From figure 9} & {}_1\eta^*=0.96 \\
\text{From figure 10} & {}_1p_b^*=-6.30 \\
\text{From figure 11} & {}_0C_{fa}Re_s=5.32 \\
 & {}_0C_{fb}Re_s=1.33 \\
\text{From figure 12} & {}_1C_{fa}=-2.79 \\
\text{and} & {}_1C_{fb}=-2.79 \\
\text{From figure 13} & {}_0q_a^*=-1.86 \\
\text{From figure 14} & {}_0q_b^*=0.59 \\
\text{From figure 2} & {}_0\tilde{\psi}(1)=4.56
\end{array}\right\} \tag{177}$$

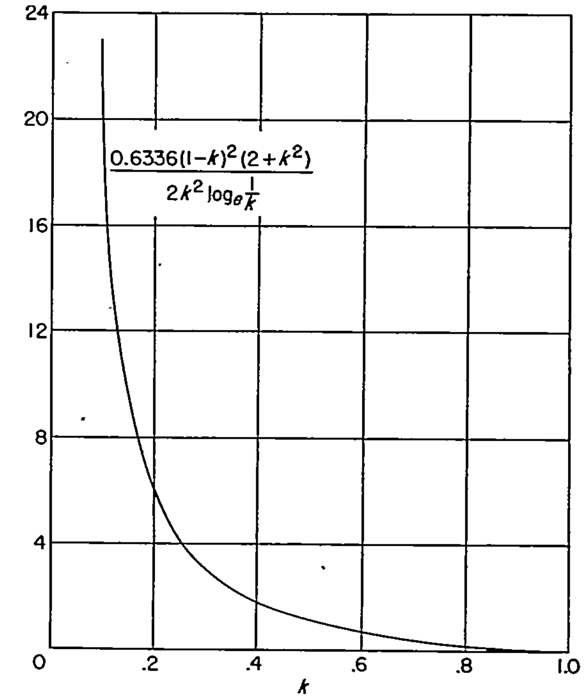


FIGURE 15.—Variation of function $0.6336(1-k)^2(2+k^2)/2k^2 \log_e \frac{1}{k}$ with diameter ratio k.

From equations (C33), (C34), and (113)

$$\left.\begin{array}{l}
{}_0\left(\dfrac{d\omega^*}{d\zeta}\right)_0=-1.85 \\
{}_0\left(\dfrac{d\omega^*}{d\zeta}\right)_1=-0.46 \\
{}_0\left(\dfrac{d^2\omega^*}{d\zeta^2}\right)_0=2.19 \\
{}_0\left(\dfrac{d^2\omega^*}{d\zeta^2}\right)_1=0.55 \\
{}_0\left(\dfrac{dT^*}{d\zeta}\right)_0=1.30 \\
{}_0\left(\dfrac{dT^*}{d\zeta}\right)_1=-0.81 \\
{}_0\left(\dfrac{d^2T^*}{d\zeta^2}\right)_0=-3.90 \\
{}_0\left(\dfrac{d^2T^*}{d\zeta^2}\right)_1=-0.98 \\
{}_0\left(\dfrac{d\log_e p^*}{d\zeta}\right)_0=3.32 \\
{}_0\left(\dfrac{d\log_e p^*}{d\zeta}\right)_1=0
\end{array}\right\} \tag{178}$$

Equation (C32) gives then

$$\left.\begin{array}{l}
{}_0\tilde{X}_a=5.46 \\
{}_0\tilde{X}_b=-0.16 \\
{}_0\tilde{Z}_a=10.0 \\
{}_0\tilde{Z}_b=-0.14
\end{array}\right\} \tag{179}$$

and equation (C31) gives

$$\left.\begin{aligned} {}_2\omega_a^* &= 17.8 \\ {}_2\omega_b^* &= -0.08 \\ {}_2T_a^* &= -27.0 \\ {}_2T_b^* &= 0.38 \end{aligned}\right\} \quad (180)$$

From equations (D9) to (D11), using equations (177) and (180),

$$\eta = 1.17(1+0.96Kn_e - 5.0Kn_e^2) \quad (181)$$

Neglecting the rarefaction correction to ${}^1\eta$ and $F(1)$, equations (135) and (136) give the friction coefficients

$$\left.\begin{aligned} Re_e C_{fa} &= 5.32(1-2.79Kn_e+17.9Kn_e^2) \\ Re_e C_{fb} &= 1.33(1-2.79Kn_e+17.9Kn_e^2) \end{aligned}\right\} \quad (182)$$

Similarly, equations (140) and (141) give the slip velocities

$$\left.\begin{aligned} \frac{u_a-U}{U} &= -2.67Kn_e+17.8Kn_e^2 \\ \frac{u_b}{U} &= 0.24Kn_e - 0.16Kn_e^2 \end{aligned}\right\} \quad (183)$$

Equations (146) and (147) give the dimensionless heat transfer

$$\left.\begin{aligned} q_a^* &= 1.86(1+10.5Kn_e-83.7Kn_e^2) \\ q_b^* &= 0.59(1-1.3Kn_e+8.2Kn_e^2) \end{aligned}\right\} \quad (184)$$

Equations (151) and (152) give the temperature jumps

$$\left.\begin{aligned} \frac{T_a-T_{wa}}{T_{wa}} &= 3.73Kn_e - 27.0Kn_e^2 \\ \frac{T_b-T_{wb}}{T_{wb}} &= 0.42Kn_e + 0.38Kn_e^2 \end{aligned}\right\} \quad (185)$$

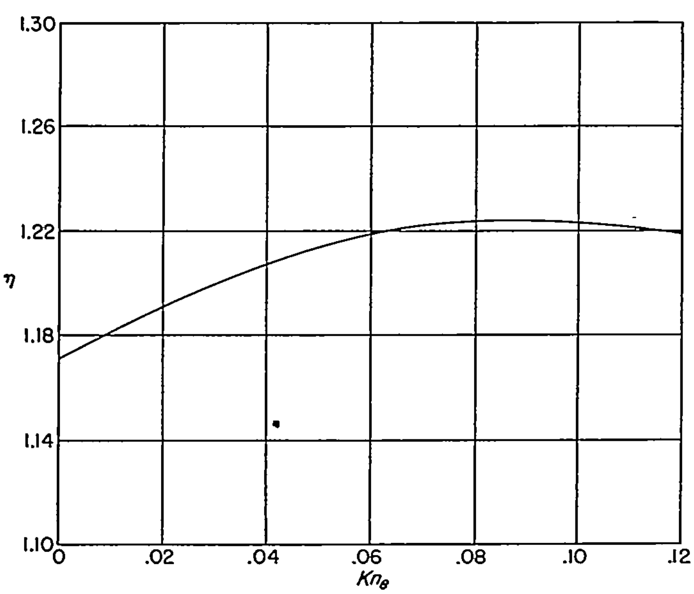


FIGURE 16.—Parameter η against effective Knudsen number. $Pr=0.715$; $\gamma=1.400$; $\beta=1$; $T_{wb}^*=1$; $\alpha=0.900$; $\sigma=1$; $k=0.5$; $M_{wa}=2$.

Equations (C28) and (154) give the pressure ratio

$$\frac{p_b}{p_a} = 2.77(1-6.3Kn_e + \ldots) \quad (186)$$

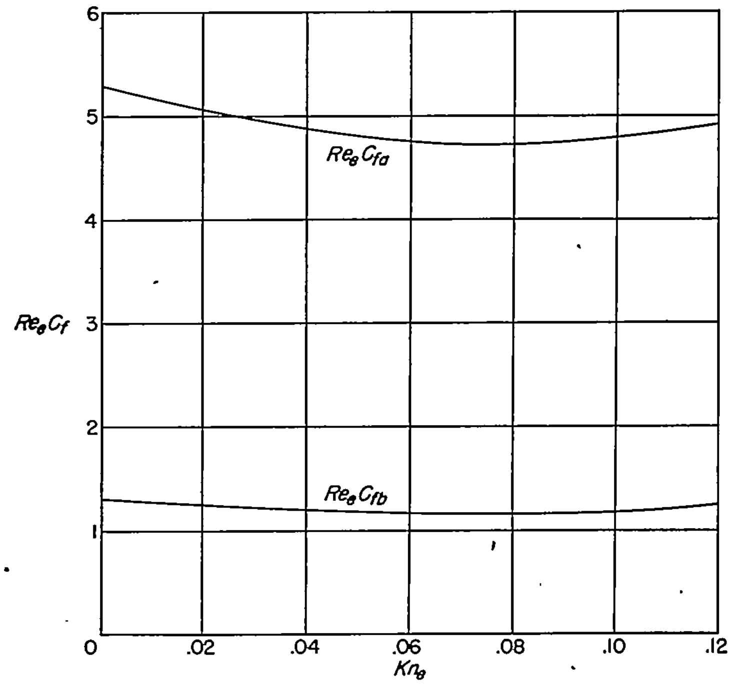


FIGURE 17.—Friction coefficient against effective Knudsen number. $Pr=0.715$; $\gamma=1.400$; $\beta=1$; $T_{wb}^*=1$; $\alpha=0.900$; $\sigma=1$; $k=0.5$; $M_{wa}=2$.

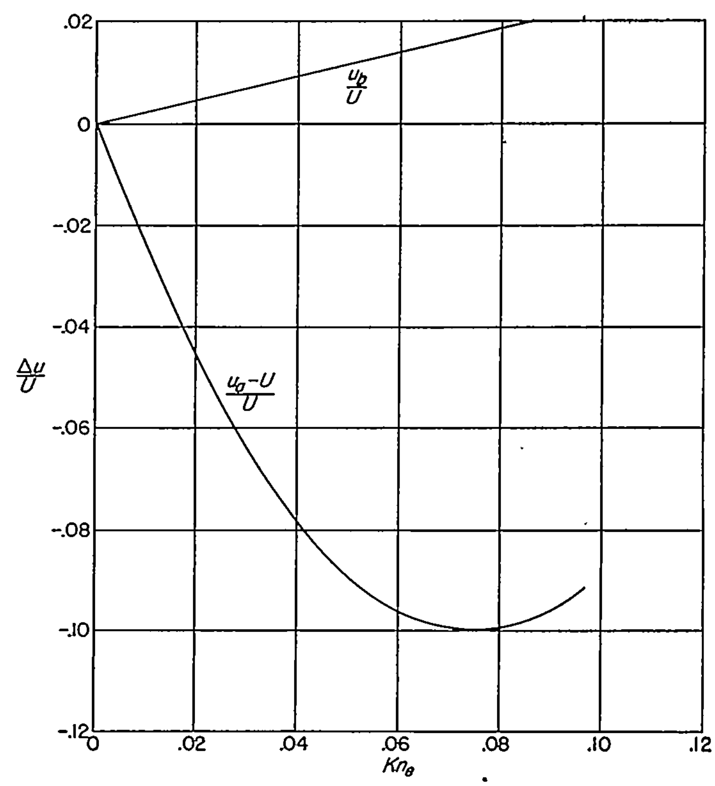


FIGURE 18.—Slip velocity against effective Knudsen number. $Pr=0.715$; $\gamma=1.400$; $\beta=1$; $T_{wb}^*=1$; $\alpha=0.900$; $\sigma=1$; $k=0.5$; $M_{wa}=2$.

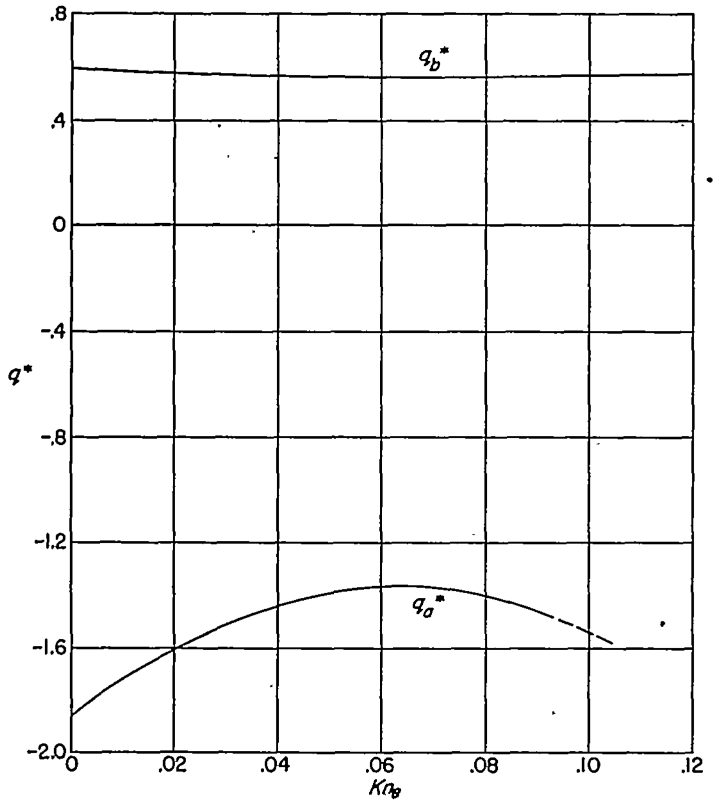


FIGURE 19.—Heat transfer against effective Knudsen number. Pr=0.715; γ=1.400; β=1; T_{wb}^*=1; α=0.900; σ=1; k=0.5; M_{wa}=2.

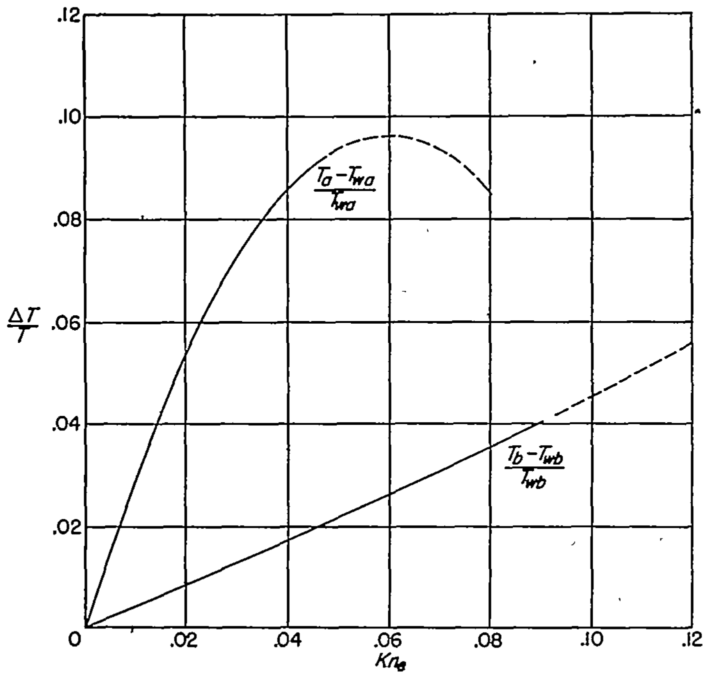


FIGURE 20.—Temperature jump against effective Knudsen number. Pr=0.715; γ=1.400; β=1; T_{wb}^*=1; α=0.900; σ=1; k=0.5; M_{wa}=2.

The variations of η, Re_eC_{fa} and Re_eC_{fb}, $(u_a-U)/U$ and u_b/U, q_a^* and q_b^*, and $(T_a-T_{wa})/T_{wa}$ and $(T_b-T_{wb})/T_{wb}$ with the effective Knudsen number Kn_e for Pr=0.715, γ=1.400, β=1, T_{wb}^*=1, α=0.900, σ=1, k=0.5, and M_{wa}=2 are given in figures 16, 17, 18, 19, and 20, respectively. Similar results could be calculated and plotted for any other choice of the parameters, but the enormous amount of work required to do so for any range of parameters does not appear to be justified at present. This is especially true in that no experimental data exist for a check.

The effort of the Burnett (second-order) terms or the Kn_e terms in equations (181) to (185) is seen to counteract the effect of the first-order slip terms or the terms in Kn_e in all cases except the temperature jump at the outer wall. The latter would show the same behavior if ${}^1\eta$ and $F(1)$ were not neglected.

SUMMARY OF RESULTS

A study was made to determine the effects of variable viscosity and thermal conductivity on the high-speed slip flow between concentric cylinders. The results are summarized as follows:

1. Satisfactory estimates of the effect of variable viscosity and thermal conductivity upon the velocity and temperature distributions were obtained from Schamberg's solution for constant values of these coefficients by basing the friction coefficient and the coefficient of heat transfer on the effective coefficients $\mu_e=\eta\mu_{wa}$ and $\lambda_e=\eta\lambda_{wa}$. These effective values of the coefficients μ_e and λ_e, in the case of equal wall temperatures, increased with the Mach number M and the "curvature" $1/k$.
2. Only the expression for the pressure ratio p_b/p_a was significantly different in this case from Schamberg's solution with constant μ because of the use of effective values of Reynolds number and Knudsen number.
3. The effect of the Burnett terms in the differential equation was more pronounced upon the pressure ratio p_b/p_a. This effect increased with the effective Knudsen number Kn_e, the Mach number M_{wa}, the temperature difference $(T_{wa}-T_{wb})/T_{wa}$, and the curvature $1/k$; it was measured by the factor ${}_0\tilde{\psi}$ of equation (123) as used in equation (124) and plotted in figures 2 and 15. The effect of the Burnett terms could best be seen by an experimental determination of the pressure ratio p_b/p_a.
4. The effect of the Burnett terms both in the differential equation and in the boundary conditions tended in all cases considered to counteract the first-order effect of slip velocity and temperature jump on the boundary.
5. For equal wall temperatures the effect of the Burnett terms was to increase the pressure ratio and to increase the skin friction on and the heat transfer to both cylinders.
6. The curvature effect on the behavior of the friction coefficient, the slip velocity, and the temperature jump as k decreased was such that they all increased at the inner cylinder and decreased at the outer cylinder. When k approaches unity, the values of all quantities reduce to those for plane Couette flow.

UNIVERSITY OF WASHINGTON,
SEATTLE, WASH., *December 20, 1951.*

APPENDIX A

SYMBOLS

A	constant of integration, equation (87)
$A_0, A_1, \ldots A_6$	numerical constants, equations (120) and (122)
$A_i = F_i - \frac{1}{\rho}\frac{\partial p}{\partial x^i}$	
A_{ij}	tensor
$\overline{A}_{ij}$	nondivergent symmetrical tensor associated with A_{ij}, equation (21)
a	adiabatic speed of sound, equation (34); determinant of metric tensor, appendix B; radius of inner cylinder
a_1	numerical constant in boundary conditions, equations (64)
a_{ij}	fundamental metric tensor, appendix B
B	constant of integration, equation (87)
b	radius of outer cylinder
b_1, b_2	numerical constants in boundary conditions, equations (64)
C	constant of integration, equation (94)
${}_1C_0, {}_1C_1$	constants of integration, equation (D25)
C_f	friction coefficient, equation (129)
$\bar{c}$	mean molecular speed, equation (38)
c_1, c_1'	numerical constants in boundary conditions, equations (64)
c_p	specific heat at constant pressure
c_v	specific heat at constant volume
D	constant of integration, equation (94)
D/Dt	convective time derivative of hydrodynamics, equation (4)
D_k	covariant derivative operator, appendix B
E	internal energy per unit mass of gas, equation (3)
$e_1, e_2, \ldots e_{10}$	numerical constants in boundary conditions, equations (64)
e_{ij}	symmetric rate of deformation tensor, appendix B
F	function, equation (102)
F_i	component of external force per unit mass, equation (2)
f	function, equation (98)
G	function, equation (104)
g	function, equation (100)
H	enthalpy per unit mass of gas, equation (9)
h	gap between cylinders, $b-a$
I	function, equation (D27)
J	function, equation (125)
$K_1, K_2, \ldots K_6$	numerical constants of stress tensor, equations (27) and (28)
Kn	Knudsen number, l/L, equation (41)
Kn_e	effective Knudsen number, equation (109)
$k = a/b$	
L	characteristic length of flow
l	mean free path of gas, equation (39)
M	Mach number, U/a, equation (35)
$m = \log_e \frac{1}{k} = \log_e \frac{b}{a}$,	equation (89)
N_1, N_2, N_3, N_4	numerical constants, equations (118)
P_{ij}	component of stress tensor, equation (2)
Pr	Prandtl number, $c_p\mu/\lambda$, equation (35)
p	static pressure of gas, equation (5)
Q	heat received per unit mass of gas, equation (8)
q_i	component of heat-flux vector, equation (3)
${}_rq_i$	rth-order approximation to q_i
$q_i^{(n)}$	nth-order correction to ${}_0q_i$, equation (14)
R	gas constant
Re	Reynolds number, equations (35) and (80)
r	radial distance from center of concentric cylinders
S	entropy per unit mass of gas, equation (9)
T	absolute temperature of gas, equation (9)
T_w	absolute temperature of wall
t	time, equation (1)
U	surface velocity of inner cylinder, $a\omega_{wa}$
u, v, w	components of macroscopic velocity in x-, y-, and z-direction, respectively
u_i	component of macroscopic velocity, equation (1)
X, Z	functions, equations (75) to (78)
x, y, z	Cartesian coordinates of physical space
x_i	component of Cartesian coordinate of physical space
α	accommodation coefficient, equation (58)
β	viscosity index, $\frac{T}{\mu}\frac{d\mu}{dT}$, equation (56)
Γ^i_{kl}	Christoffel symbol of second kind, appendix B
γ	ratio of specific heats of gas, c_p/c_v
Δ	function, equation (93)
δ_{ij}	unit tensor, equation (5)
$\epsilon = 1-k^2$,	equation (D18)

ζ	variable, equation (90)
η	constant, equation (91)
Θ	divergence of velocity, appendix B
$\theta_1, \theta_2, \ldots \theta_5$	numerical constants in heat-flux vector, equations (29) and (30)
λ	coefficient of thermal conductivity, equation (20)
μ	coefficient of viscosity, equation (19)
ξ	variable, equation (89)
ρ	density of gas, equation (1)
σ	Maxwell's reflection coefficient, equation (57)
τ	shearing stress
τ_{ij}	component of stress tensor, equation (5)
${}_r\tau_{ij}$	rth-order approximation to τ_{ij}
$\tau_{ij}^{(n)}$	nth-order correction to ${}_0\tau_{ij}$, equation (13)
Φ	dissipation function, equation (24)
ϕ	angular coordinate in cylindrical coordinates
${}_0\psi$	function, equation (115)
${}_0\tilde{\psi}$	function, ${}_0\psi/{}_0\eta^2$, equation (121)
ω_{wa}	angular velocity of inner cylinder
ω^*	dimensionless angular velocity, equation (95)
$\bar{\omega}$	numerical constant, equation (C5)
Subscripts:	
$0, 1, 2, \ldots$	in front of any symbol, order of approximation to boundary conditions, equations (C27) and (C28)
$\alpha, \beta, i, j, k, l, m$	covariant vector and tensor indices, appendix B
e	effective value
wa	conditions at wall temperature T_{wa}
$\vert$	indicates covariant differentiation, appendix B
Superscripts:	
$0, 1, 2, \ldots$	in front of any symbol, order of approximation to solution of differential equation, equation (110)
$\alpha, \beta, i, j, k, l, m$	contravariant vector and tensor indices, appendix B
β	exponent in viscosity-temperature relation

APPENDIX B

TRANSFORMATION OF EQUATIONS OF MOTION TO POLAR COORDINATES

The equations of fluid mechanics have been written in Cartesian tensor form in this report as in references 1 and 3. However, the problem considered had to be set up in cylindrical coordinates and Cartesian coordinates were abandoned in the process. In order to take into account all possible systems of curvilinear coordinates, it is best to express the equations of fluid mechanics, including the Burnett terms, in general tensor form for any space with a metric form of the type (ref. 23)

$$ds^2 = a_{ij}dx^i dx^j$$

By the principle of covariance all that one has to do is to express each term in the equations as the proper invariant (scalar, vector, or tensor), which reduces to the known form when the coordinates are Cartesian. The method and notation used are to be found in reference 23. The distinction between superscripts and subscripts is necessary.

There is no difficulty with the equation of continuity. It is almost in the form of a scalar equation already. Redefine the comoving time derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u^k D_k$$

where D_k is now the covariant derivative operator. For any scalar such as ρ the covariant derivative is the same as the ordinary partial derivative. The divergence of the velocity is the scalar

$$\Theta \equiv D_k u^k \equiv u^k{}_{|k} = \frac{\partial u^k}{\partial x^k} + \Gamma^k_{km}\, u^m = \frac{1}{\sqrt{a}}\frac{\partial}{\partial x^k}(\sqrt{a}\, u^k)$$

In other words, take the covariant derivative of the contravariant velocity vector u^k and contract the result; $a = |a_{ij}|$, the determinant of the metric tensor a_{ij}. By definition the covariant derivative is

$$u^k{}_{|l} = \frac{\partial u^k}{\partial x^l} + \Gamma^k_{lm}\, u^m$$

where the terms Γ^k_{lm} are the Christoffel symbols of the second kind. The continuity equation is then

$$\left.\begin{aligned} &\frac{D\rho}{Dt} + \rho\Theta = 0 \\ \text{or}\qquad &\frac{\partial \rho}{\partial t} + (\rho u^k)_{|k} = 0 \end{aligned}\right\} \tag{B1}$$

for, when covariant differentiation becomes ordinary differentiation, these expressions reduce to the known equation, equation (1) (ref. 23, p. 196).

Similarly, the momentum equation is a vector equation and the energy equation is a scalar equation. All that is needed to give them general tensor form is to place the indices properly and to use covariant differentiation. They are obviously

$$\rho\frac{Du_k}{Dt}-\rho F_k+\frac{\partial p}{\partial x^k}+\tau_k{}^{\alpha}{}_{|\alpha}=0 \tag{B2}$$

$$\rho\frac{DE}{Dt}+p\Theta+\tau^{\alpha\beta}u_{\alpha|\beta}+q^{\alpha}{}_{|\alpha}=0 \tag{B3}$$

which replace equations (6) and (7).

Expansions (13) and (14) remain the same as long as it is borne in mind that each $\tau_{ij}{}^{(n)}$ is a covariant tensor and $q_i{}^{(n)}$ is a covariant vector. Equations (19) and (20) for the second-order approximations remain the same:

$${}_1\tau_{ij}=-2\mu\overline{\frac{\partial u_i}{\partial x^j}}$$

$${}_1q_i=-\lambda\frac{\partial T}{\partial x^i}$$

provided $\overline{\frac{\partial u_i}{\partial x^j}}$ is redefined as the covariant tensor

$$\overline{\frac{\partial u_i}{\partial x^j}}\equiv e_{ij}=\frac{1}{2}\left(u_{i|j}+u_{j|i}\right)-\frac{1}{3}\Theta a_{ij}$$

One must use a_{ij} here as it is the fundamental tensor of the space; a^{ij} is the fundamental contravariant tensor, and δ_i^j is the fundamental mixed tensor.

By continuing this process the expressions for the third-order Burnett terms become

$$\begin{aligned}\tau_{ij}{}^{(2)}=&K_1\frac{\mu^2}{p}\Theta e_{ij}+K_2\frac{\mu^2}{p}\left[\frac{1}{2}\left(A_{i|j}+A_{j|i}\right)-\frac{1}{3}A^{\alpha}{}_{|\alpha}a_{ij}-\frac{1}{2}\left(u^m{}_{|i}u_{j|m}+u^m{}_{|j}u_{i|m}\right)+\frac{1}{3}u^k{}_{|m}u^m{}_{|k}a_{ij}-\left(e_i{}^m u_{m|j}+e_j{}^m u_{m|i}\right)+\frac{2}{3}e^{km}u_{k|m}a_{ij}\right]+\\&K_3\frac{\mu^2}{\rho T}\left[T_{|ij}-\frac{1}{3\sqrt{a}}\frac{\partial}{\partial x^k}\left(a^{km}\sqrt{a}\frac{\partial T}{\partial x^m}\right)a_{ij}\right]+K_4\frac{\mu^2}{\rho p T}\left[\frac{1}{2}\left(p_{|i}T_{|j}+p_{|j}T_{|i}\right)-\frac{1}{3}a^{km}p_{|k}T_{|m}a_{ij}\right]+K_5\frac{\mu^2}{\rho T^2}\left(T_{|i}T_{|j}-\frac{1}{3}a^{km}T_{|k}T_{|m}a_{ij}\right)+\\&K_6\frac{\mu^2}{p}\left[\frac{1}{2}\left(e_i{}^{\alpha}e_{\alpha j}+e_j{}^{\alpha}e_{\alpha i}\right)-\frac{1}{3}e^{\alpha\beta}e_{\alpha\beta}a_{ij}\right]\end{aligned}$$

$$q_i{}^{(2)}=\theta_1\frac{\mu^2}{\rho T}\Theta T_{|i}+\theta_2\frac{\mu^2}{\rho T}\left[\frac{2}{3}\frac{\partial}{\partial x^i}\left(\Theta T\right)+2u^k{}_{|i}T_{|k}\right]+\theta_3\frac{\mu^2}{\rho p}p_{|k}e^k{}_i+\theta_4\frac{\mu^2}{\rho}e^k{}_{i|k}+\theta_5\frac{\mu^2}{\rho T}T_{|k}e^k{}_i$$

where the covariant vector $A_i=F_i-\frac{1}{\rho}\frac{\partial p}{\partial x^i}$, $T_{|i}=\frac{\partial T}{\partial x^i}$, $p_{|i}=\frac{\partial p}{\partial x^i}$, and

$$T_{|ij}=\frac{\partial^2 T}{\partial x^j\,\partial x^i}-\Gamma^k_{ij}\frac{\partial T}{\partial x^k}$$

The Cartesian symbol $\overline{A}_{ij}$ has been replaced by the covariant expression

$$\overline{A}_{ij}=\frac{1}{2}\left(A_{ij}+A_{ji}\right)-\frac{1}{3}A^k{}_k a_{ij}$$

In order to obtain equations (54) to 56), it is necessary to introduce plane polar coordinates $r=x^1$ and $\phi=x^2$. The metric form is

$$ds^2=dr^2+r^2\,d\phi^2$$

so $a_{11}=a^{11}=1$, $a_{22}=r^2=\frac{1}{a^{22}}$, and all other values of a_{ij} are zero while $a=r^2$. The only nonvanishing Christoffel symbols are

$$\Gamma^1_{22}=-r$$

$$\Gamma^2_{12}=\Gamma^2_{21}=\frac{1}{r}$$

Denote the physical components of the velocity vector by u_r and u_ϕ, where $u_r=u_1=u^1$ and $u_\phi=ru^2=\frac{u_2}{r}$. Similarly, the physical components of the vector q_i are q_r and q_ϕ and, of the stress tensor (ref. 23, pp. 145 to 146),

$$\tau_{rr}=\tau_{11}=\tau^{11}$$

$$\tau_{r\phi}=\frac{1}{r}\tau_{12}=r\tau^{12}$$

$$\tau_{\phi\phi}=\frac{1}{r^2}\tau_{22}=r^2\tau^{22}$$

Then

$$\begin{aligned}\Theta&=\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r\right)+\frac{1}{r}\frac{\partial u_\phi}{\partial \phi}\\&=\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\phi}{\partial \phi}\end{aligned} \tag{B4}$$

and the equation of continuity becomes

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho u_r\right)+\frac{1}{r}\frac{\partial}{\partial \phi}\left(\rho u_\phi\right)=0 \tag{B5}$$

Written out, equations (B2) and (B3) become equations (31) and (32) in covariant form, $D^k=a^{k\alpha}D_\alpha$,

$$\rho\frac{Du_k}{Dt}-\rho F_k+p_{|k}-2D^{\alpha}(\mu e_{k\alpha})+D^{\alpha}\tau_{k\alpha}^{(2)}=0 \qquad \text{(B6)}$$

$$\rho\frac{DE}{Dt}+p\Theta-2\mu e^{\alpha\beta}u_{\beta|\alpha}-D^{\alpha}(\lambda T_{|\alpha})+\tau^{(2)\alpha\beta}u_{\alpha|\beta}+q^{(2)\alpha}{}_{|\alpha}=0 \qquad \text{(B7)}$$

The components of $\frac{Du_k}{Dt}$ become, using covariant derivatives and the nonvanishing Christoffel symbols,

$$\frac{Du_1}{Dt}=\frac{\partial u_1}{\partial t}+u^k u_{1|k}=\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\phi}{r}\frac{\partial u_r}{\partial\phi}-\frac{u_\phi^2}{r}$$

$$\frac{Du_2}{Dt}=\frac{\partial u_2}{\partial t}+u^k u_{2|k}=r\frac{\partial u_\phi}{\partial t}+u_r\frac{\partial}{\partial r}(ru_\phi)+u_\phi\frac{\partial u_\phi}{\partial\phi}$$

Let $F_1=F_r$, $F_2=rF_\phi$, $D_1p=\frac{\partial p}{\partial r}$, and $D_2p=\frac{\partial p}{\partial\phi}$. The covariant components of e_{ij} are

$$e_{11}=\frac{\partial u_r}{\partial r}-\frac{1}{3}\Theta$$

$$e_{12}=e_{21}=\frac{1}{2}\left(\frac{\partial u_r}{\partial\phi}+r\frac{\partial u_\phi}{\partial r}-u_\phi\right)$$

$$e_{22}=r\left(\frac{\partial u_\phi}{\partial\phi}+u_r\right)-\frac{1}{3}\Theta r^2$$

while the contravariant ones are

$$e^{11}=\frac{\partial u_r}{\partial r}-\frac{1}{3}\Theta$$

$$e^{12}=e^{21}=\frac{1}{2}\left(\frac{1}{r^2}\frac{\partial u_r}{\partial\phi}+\frac{1}{r}\frac{\partial u_\phi}{\partial r}-\frac{u_\phi}{r^2}\right)$$

$$e^{22}=\frac{1}{r^3}\left(\frac{\partial u_\phi}{\partial\phi}+u_r\right)-\frac{1}{3}\frac{1}{r^2}\Theta$$

Since

$$u_{1|1}=\frac{\partial u_r}{\partial r}$$

$$u_{1|2}=\frac{\partial u_r}{\partial\phi}-u_\phi$$

$$u_{2|1}=r\frac{\partial u_\phi}{dr}$$

$$u_{2|2}=r\left(\frac{\partial u\phi}{\partial\phi}+u_r\right)$$

it follows that in equation (B7) the dissipation term is

$$e^{kl}u_{k|l}=\left(\frac{\partial u_r}{\partial r}\right)^2+\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\phi}+\frac{\partial u_\phi}{\partial r}-\frac{u_\phi}{r}\right)^2+\left(\frac{1}{r}\frac{\partial u_\phi}{\partial\phi}+\frac{u_r}{r}\right)^2-\frac{1}{3}\Theta^2$$

Consequently, the equations of motion and energy in two-dimensional polar coordinates are

$$\rho\left(\frac{\partial u_r}{dt}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\phi}{r}\frac{\partial u_r}{\partial\phi}-\frac{u_\phi^2}{r}\right)-\rho F_r+\frac{\partial p}{\partial r}-2\frac{\partial}{\partial r}\left[\mu\left(\frac{\partial u_r}{\partial r}-\frac{1}{3}\Theta\right)\right]-\frac{1}{r}\frac{\partial}{\partial\phi}\left[\mu\left(\frac{1}{r}\frac{\partial u_r}{\partial\phi}+\frac{\partial u_\phi}{\partial r}-\frac{u_\phi}{r}\right)\right]-\frac{2\mu}{r}\left(\frac{\partial u_r}{\partial r}-\frac{1}{r}\frac{\partial u_\phi}{\partial\phi}-\frac{u_r}{r}\right)+D^{\alpha}\tau_{1\alpha}^{(2)}=0 \qquad \text{(B8)}$$

$$\rho\left(\frac{\partial u_\phi}{\partial t}+u_r\frac{\partial u_\phi}{\partial r}+\frac{u_\phi}{r}\frac{\partial u_\phi}{\partial\phi}+\frac{u_\phi u_r}{r}\right)-\rho F_\phi+\frac{1}{r}\frac{\partial p}{\partial\phi}-\frac{\partial}{\partial r}\left[\mu\left(\frac{1}{r}\frac{\partial u_r}{\partial\phi}+\frac{\partial u_\phi}{\partial r}-\frac{u_\phi}{r}\right)\right]-\frac{2}{r}\frac{\partial}{\partial\phi}\left[\mu\left(\frac{1}{r}\frac{\partial u_\phi}{\partial\phi}+\frac{u_r}{r}-\frac{1}{3}\Theta\right)\right]-\frac{2\mu}{r}\left(\frac{1}{r}\frac{\partial u_r}{\partial\phi}+\frac{\partial u_\phi}{\partial r}-\frac{u_\phi}{r}\right)+D^{\alpha}\tau_{2\alpha}^{(2)}=0 \qquad \text{(B9)}$$

$$\rho\left(\frac{\partial E}{\partial t}+u_r\frac{\partial E}{\partial r}+\frac{u_\phi}{r}\frac{\partial E}{\partial\phi}\right)+p\Theta-2\mu\left[\left(\frac{\partial u_r}{\partial r}\right)^2+\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\phi}+\frac{\partial u_\phi}{\partial r}-\frac{u_\phi}{r}\right)^2+\left(\frac{1}{r}\frac{\partial u_\phi}{\partial\phi}+\frac{u_r}{r}\right)^2-\frac{1}{3}\Theta^2\right]-\frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda\frac{\partial T}{\partial r}\right)-\frac{1}{r^2}\frac{\partial}{\partial\phi}\left(\lambda\frac{\partial T}{\partial\phi}\right)+\tau^{(2)kl}u_{k|l}+q^{(2)k}{}_{|k}=0 \qquad \text{(B10)}$$

For $\Theta=0$ and $\mu=$Constant these reduce to the well-known equations for an incompressible fluid (refs. 12 and 13). It still remains to find the components of $q_k^{(2)}$ and $\tau_{kl}^{(2)}$ and substitute into these equations. For the former the result is immediate:

$$q_r^{(2)}=\theta_1\frac{\mu^2}{\rho T}\Theta\frac{\partial T}{\partial r}+\theta_2\frac{\mu^2}{\rho T}\left[\frac{2}{3}\frac{\partial}{\partial r}(\Theta T)+2\frac{\partial u_r}{\partial r}\frac{\partial T}{\partial r}+\frac{2}{r}\frac{\partial u_\phi}{\partial r}\frac{\partial T}{\partial\phi}\right]+\theta_3\frac{\mu^2}{\rho p}\left[\left(\frac{\partial u_r}{\partial r}-\frac{1}{3}\Theta\right)\frac{\partial p}{\partial r}+\frac{1}{2r}\frac{\partial p}{\partial\phi}\left(\frac{1}{r}\frac{\partial u_r}{\partial\phi}+\frac{\partial u_\phi}{\partial r}-\frac{u_\phi}{r}\right)\right]+\theta_4\frac{\mu^2}{\rho}\left[\frac{\partial^2u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{1}{r^2}\frac{\partial u_\phi}{\partial\phi}-\frac{u_r}{r^2}-\frac{1}{3}\frac{\partial}{\partial r}\Theta\right]+\theta_5\frac{\mu^2}{\rho T}\left[\left(\frac{\partial u_r}{\partial r}-\frac{1}{3}\Theta\right)\frac{\partial T}{\partial r}+\frac{1}{2r}\frac{\partial T}{\partial\phi}\left(\frac{1}{r}\frac{\partial u_r}{\partial\phi}+\frac{\partial u_\phi}{\partial r}-\frac{u_\phi}{r}\right)\right]$$

$$q_\phi^{(2)}=\theta_1\frac{\mu^2}{\rho T}\Theta\frac{1}{r}\frac{\partial T}{\partial\phi}+\theta_2\frac{\mu^2}{\rho T}\left[\frac{2}{3}\frac{1}{r}\frac{\partial}{\partial\phi}(\Theta T)+2\left(\frac{1}{r}\frac{\partial u_r}{\partial\phi}-\frac{u_\phi}{r}\right)\frac{\partial T}{\partial r}+\frac{2}{r}\frac{\partial T}{\partial\phi}\left(\frac{1}{r}\frac{\partial u_\phi}{\partial\phi}+\frac{u_r}{r}\right)\right]+\theta_3\frac{\mu^2}{\rho p}\left[\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\phi}+\frac{\partial u_\phi}{\partial r}-\frac{u_\phi}{r}\right)\frac{\partial p}{\partial r}-\frac{1}{3r}\frac{\partial p}{\partial\phi}\Theta+\left(\frac{1}{r}\frac{\partial u_\phi}{\partial\phi}+\frac{u_r}{r}\right)\frac{1}{r}\frac{\partial p}{\partial\phi}\right]+\theta_4\frac{\mu^2}{\rho}\left[\frac{1}{2}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u_r}{\partial\phi}+\frac{\partial u_\phi}{\partial r}-\frac{u_\phi}{r}\right)+\frac{1}{r}\left(\frac{1}{r}\frac{\partial u_r}{\partial\phi}+\frac{\partial u_\phi}{\partial r}-\frac{u_\phi}{r}\right)\right]+\theta_5\frac{\mu^2}{\rho T}\left[\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\phi}+\frac{\partial u_\phi}{\partial r}-\frac{u_\phi}{r}\right)\frac{\partial T}{\partial r}+\left(\frac{1}{r}\frac{\partial u_\phi}{\partial\phi}+\frac{u_r}{r}\right)\frac{1}{r}\frac{\partial T}{\partial\phi}-\frac{1}{3r}\frac{\partial T}{\partial\phi}\Theta\right]$$

A longer, more tedious calculation leads to the stress components

$$\tau_{rr}^{(2)}=K_1\frac{\mu^2}{p}\Theta\left(\frac{\partial u_r}{\partial r}-\frac{1}{3}\Theta\right)+K_2\frac{\mu^2}{p}\left[\frac{2}{3}\frac{\partial A_1}{\partial r}-\frac{1}{3}\frac{A_1}{r}-\frac{1}{3r^2}\frac{\partial A_2}{\partial \phi}-\frac{2}{3}\left(\frac{\partial u_\phi}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial u_\phi}{\partial \phi}+\frac{u_r}{r}\right)^2+\frac{1}{3}\left(\frac{1}{r}\frac{\partial u_r}{\partial \phi}-\frac{u_\phi}{r}\right)^2-2\left(\frac{\partial u_r}{\partial r}\right)^2-\right.$$
$$\left.\frac{2}{3}\frac{\partial u_\phi}{\partial r}\left(\frac{1}{r}\frac{\partial u_r}{\partial \phi}-\frac{u_\phi}{r}\right)+\frac{2}{3}\Theta\left(\frac{\partial u_r}{\partial r}-\frac{1}{3}\Theta\right)\right]+K_6\frac{\mu^2}{p}\left[\frac{1}{3}\left(\frac{\partial u_r}{\partial r}\right)^2-\frac{1}{9}\Theta^2+\frac{1}{12}\left(\frac{1}{r}\frac{\partial u_r}{\partial \phi}+\frac{\partial u_\phi}{\partial r}-\frac{u_\phi}{r}\right)^2\right]+K_3\frac{\mu^2}{p}R\left(\frac{2}{3}\frac{\partial^2 T}{\partial r^2}-\frac{1}{3r}\frac{\partial T}{\partial r}-\right.$$
$$\left.\frac{1}{3r^2}\frac{\partial^2 T}{\partial \phi^2}\right)+K_4\frac{\mu^2}{p\rho T}\left(\frac{2}{3}\frac{\partial p}{\partial r}\frac{\partial T}{\partial r}-\frac{1}{3r^2}\frac{\partial p}{\partial \phi}\frac{\partial T}{\partial \phi}\right)+K_5\frac{\mu^2}{pT}R\left[\frac{2}{3}\left(\frac{\partial T}{\partial r}\right)^2-\frac{1}{3}\left(\frac{1}{r}\frac{\partial T}{\partial \phi}\right)^2\right]$$

$$\tau_{\phi\phi}^{(2)}=K_1\frac{\mu^2}{p}\Theta\left(\frac{1}{r}\frac{\partial u_\phi}{\partial \phi}+\frac{u_r}{r}-\frac{1}{3}\Theta\right)+K_2\frac{\mu^2}{p}\left(\frac{2}{3r^2}\frac{\partial A_2}{\partial \phi}+\frac{2}{3}\frac{A_1}{r}-\frac{1}{3}\frac{\partial A_1}{\partial r}\right)+K_2\frac{\mu^2}{p}\left[\frac{1}{3}\left(\frac{\partial u_\phi}{\partial r}-\frac{1}{r}\frac{\partial u_r}{\partial \phi}+\frac{u_\phi}{r}\right)^2-\left(\frac{1}{r}\frac{\partial u_r}{\partial \phi}-\frac{u_\phi}{r}\right)^2+\right.$$
$$\left.\frac{\partial u_r}{\partial r}\left(\frac{\partial u_r}{\partial r}-\frac{2}{3}\Theta\right)+\frac{4}{9}\Theta^2-2\left(\frac{1}{r}\frac{\partial u_\phi}{\partial \phi}+\frac{u_r}{r}\right)^2\right]+K_6\frac{\mu^2}{p}\left[\frac{1}{12}\left(\frac{1}{r}\frac{\partial u_\phi}{\partial \phi}+\frac{\partial u_\phi}{\partial r}-\frac{u_\phi}{r}\right)^2-\frac{1}{9}\left(\frac{\partial u_r}{\partial r}\right)^2-\frac{2}{9}\frac{\partial u_r}{\partial r}\left(\frac{1}{r}\frac{\partial u_\phi}{\partial \phi}+\frac{u_r}{r}\right)+\right.$$
$$\left.\frac{2}{9}\left(\frac{1}{r}\frac{\partial u_\phi}{\partial \phi}+\frac{u_r}{r}\right)^2\right]+K_3\frac{\mu^2}{p}R\left(\frac{2}{3}\frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2}+\frac{2}{3}\frac{1}{r}\frac{\partial T}{\partial r}-\frac{1}{3}\frac{\partial^2 T}{\partial r^2}\right)+K_4\frac{\mu^2}{p\rho T}\left(\frac{2}{3}\frac{1}{r^2}\frac{\partial p}{\partial \phi}\frac{\partial T}{\partial \phi}-\frac{1}{3}\frac{\partial p}{\partial r}\frac{\partial T}{\partial r}\right)+K_5\frac{\mu^2}{pT}R\left[\frac{2}{3}\left(\frac{1}{r}\frac{\partial T}{\partial \phi}\right)^2-\frac{1}{3}\left(\frac{\partial T}{\partial r}\right)^2\right]$$

$$\tau_{r\phi}^{(2)}=\tau_{\phi r}^{(2)}=\frac{1}{2}K_1\frac{\mu^2}{p}\Theta\left(\frac{1}{r}\frac{\partial u_r}{\partial \phi}+\frac{\partial u_\phi}{\partial r}-\frac{u_\phi}{r}\right)+K_2\frac{\mu^2}{p}\left[\frac{1}{2}\left(\frac{1}{r}\frac{\partial A_1}{\partial \phi}+\frac{1}{r}\frac{\partial A_2}{\partial r}\right)-\frac{A_2}{r^2}-\frac{2}{3}\Theta\left(\frac{1}{r}\frac{\partial u_r}{\partial \phi}+\frac{\partial u_\phi}{\partial r}-\frac{u_\phi}{r}\right)-\frac{\partial u_r}{\partial r}\left(\frac{1}{r}\frac{\partial u_r}{\partial \phi}-\frac{u_\phi}{r}\right)-\right.$$
$$\left.\frac{\partial u_\phi}{\partial r}\left(\frac{1}{r}\frac{\partial u_\phi}{\partial \phi}+\frac{u_r}{r}\right)\right]+K_3\frac{\mu^2}{p}R\frac{1}{r}\frac{\partial}{\partial \phi}\left(\frac{\partial T}{\partial r}-\frac{T}{r}\right)+K_4\frac{\mu^2}{p\rho T}\frac{1}{2}\left(\frac{1}{r}\frac{\partial p}{\partial r}\frac{\partial T}{\partial \phi}+\frac{1}{r}\frac{\partial p}{\partial \phi}\frac{\partial T}{\partial r}\right)+K_5\frac{\mu^2}{p}\frac{R}{T}\frac{1}{r}\frac{\partial T}{\partial \phi}\frac{\partial T}{\partial r}+$$
$$\frac{1}{6}K_6\frac{\mu^2}{p}\Theta\left(\frac{1}{r}\frac{\partial u_r}{\partial \phi}+\frac{\partial u_\phi}{\partial r}-\frac{u_\phi}{r}\right)$$

Since for the symmetrical case considered in this report $u_r=0$ and all partial derivatives with respect to ϕ vanish, $\Theta=0, A_1=-\frac{1}{\rho}\frac{dp}{dr}$, and $A_2=0$, it follows that these expressions simplify considerably, becoming simply

$$q_r^{(2)}=0$$

$$q_\phi^{(2)}=-2\theta_2\frac{\mu^2}{\rho T}\frac{u_\phi}{r}\frac{dT}{dr}+\frac{1}{2}\theta_3\frac{\mu^2}{p\rho}\frac{dp}{dr}\left(\frac{du_\phi}{dr}-\frac{u_\phi}{r}\right)+\frac{1}{2}\theta_4\frac{\mu^2}{\rho}\frac{d}{dr}\left(\frac{du_\phi}{dr}-\frac{u_\phi}{r}\right)+\theta_4\frac{\mu^2}{\rho r}\left(\frac{du_\phi}{dr}-\frac{u_\phi}{r}\right)+\frac{1}{2}\theta_5\frac{\mu^2}{\rho T}\frac{dT}{dr}\left(\frac{du_\phi}{dr}-\frac{u_\phi}{r}\right)$$

$$\tau_{rr}^{(2)}=\frac{1}{3}K_2\frac{\mu^2}{p}\left[\frac{1}{\rho r}\frac{dp}{dr}-2\frac{d}{dr}\left(\frac{1}{\rho}\frac{dp}{dr}\right)-2\left(\frac{du_\phi}{dr}\right)^2+\left(\frac{u_\phi}{r}\right)^2+2\frac{u_\phi}{r}\frac{du_\phi}{dr}\right]+\frac{1}{12}K_6\frac{\mu^2}{p}\left(\frac{du_\phi}{dr}-\frac{u_\phi}{r}\right)^2+\frac{1}{3}K_3\frac{\mu^2}{p}R\left(2\frac{d^2T}{dr^2}-\frac{1}{r}\frac{dT}{dr}\right)+\frac{2}{3}K_4\frac{\mu^2}{p\rho T}\frac{dp}{dr}\frac{dT}{dr}+\frac{2}{3}K_5\frac{\mu^2}{pT}R\left(\frac{dT}{dr}\right)^2 \tag{B11}$$

$$\tau_{\phi\phi}^{(2)}=\frac{1}{3}K_2\frac{\mu^2}{p}\left[\frac{d}{dr}\left(\frac{1}{\rho}\frac{dp}{dr}\right)-\frac{2}{\rho r}\frac{dp}{dr}+\left(\frac{du_\phi}{dr}+\frac{u_\phi}{r}\right)^2-3\left(\frac{u_\phi}{r}\right)^2\right]+\frac{1}{12}K_6\frac{\mu^2}{p}\left(\frac{du_\phi}{dr}-\frac{u_\phi}{r}\right)^2+\frac{1}{3}K_3\frac{\mu^2}{p}R\left(\frac{2}{r}\frac{dT}{dr}-\frac{d^2T}{dr^2}\right)-\frac{1}{3}K_4\frac{\mu^2}{p\rho T}\frac{dp}{dr}\frac{dT}{dr}-\frac{1}{3}K_5\frac{\mu^2}{pT}R\left(\frac{dT}{dr}\right)^2 \tag{B12}$$

$$\tau_{r\phi}^{(2)}=\tau_{\phi r}^{(2)}=0$$

Equations (B8), (B9), and B10) reduce for this symmetric case to

$$-\rho\frac{u_\phi^2}{r}+\frac{dp}{dr}+\frac{d\tau_{rr}^{(2)}}{dr}+\frac{1}{r}(\tau_{rr}^{(2)}-\tau_{\phi\phi}^{(2)})=0$$

$$\frac{d}{dr}\left[\mu\left(\frac{du_\phi}{dr}-\frac{u_\phi}{r}\right)\right]+\frac{2}{r}\mu\left(\frac{du_\phi}{dr}-\frac{u_\phi}{r}\right)=0$$

$$\mu\left(\frac{du_\phi}{dr}-\frac{u_\phi}{r}\right)^2+\frac{1}{r}\frac{d}{dr}\left(r\lambda\frac{dT}{dr}\right)=0$$

But $\tau_{rr}=\tau_{rr}^{(2)}$, $\tau_{r\phi}=\tau_{r\phi}^{(1)}=-\mu\left(\frac{du_\phi}{dr}-\frac{u_\phi}{r}\right)$, $\tau_{\phi\phi}=\tau_{\phi\phi}^{(2)}$, $q_r=q_r^{(1)}=-\lambda\frac{dT}{dr}$, and $q_\phi=q_\phi^{(2)}$ in this case, so these are precisely equations (47) to (49) while equations (B12) and (B11) agree with equations (51) and (52). No second-order terms occur in equations (50) and (53) since $\tau_{r\phi}^{(2)}=0$ and $q_r^{(2)}=0$. The second-order term for $q_\phi^{(2)}$ does not occur because its derivative with respect to ϕ is zero in the problem considered.

APPENDIX C

REDUCTION OF BOUNDARY CONDITIONS TO DIMENSIONLESS FORM

Making use of equations (79) to (81), the boundary conditions, equations (71) to (74), can be written in dimensionless form as follows:

$$u_a^*=1+a_1\frac{\mu_a^*}{(T_a^*)^{1/2}}\left[r^*\frac{d}{dr^*}\left(\frac{u^*}{r^*}\right)\right]_a\bar{\omega}+\frac{(\mu_a^*)^2}{(T_a^*)^2}X_a^*\bar{\omega}^2 \quad \text{(C1)}$$

$$u_b^*=0-a_1\frac{(T_b^*)^{1/2}\mu_b^*}{T_a^*p_b^*}\left[r^*\frac{d}{dr^*}\left(\frac{u^*}{r^*}\right)\right]_b\bar{\omega}+\frac{(\mu_b^*)^2}{(p_b^*)^2(T_a^*)^2}X_b^*\bar{\omega}^2 \quad \text{(C2)}$$

$$T_a^*=1+c_1\frac{\mu_a^*}{(T_a^*)^{1/2}}\left(\frac{dT^*}{dr^*}\right)_a\bar{\omega}+\frac{(\mu_a^*)^2}{(T_a^*)^2}Z_a^*\bar{\omega}^2 \quad \text{(C3)}$$

$$T_b^*=T_{wb}^*-c_1\frac{(T_b^*)^{1/2}\mu_b^*}{T_a^*p_b^*}\left(\frac{dT^*}{dr^*}\right)_b\bar{\omega}+\frac{(\mu_b^*)^2}{(p_b^*)^2(T_a^*)^2}Z_b^*\bar{\omega}^2 \quad \text{(C4)}$$

where

$$\bar{\omega}=\frac{R^{1/2}\mu_{wa}T_a}{ap_a(T_{wa})^{1/2}}=\frac{(1-k)\gamma^{1/2}}{k}\frac{M_{wa}}{Re_{wa}}=0.796\left(\frac{1-k}{k}\right)Kn_{wa} \quad \text{(C5)}$$

using equation (41).

The functions X_a^*, X_b^*, Z_a^*, and Z_b^* are obtained from equations (75) to (78) by use of equations (80) and (81) and are

$$\left.\begin{aligned}
X_a^*&\equiv\frac{a^2X_a}{RT_{wa}U}\\
&=-\frac{5}{6}T_a^*\left\{\frac{d}{dr^*}\left[r^*\frac{d}{dr^*}\left(\frac{u^*}{r^*}\right)\right]\right\}_a-\left(5.167-\frac{8}{15}\right)\left(\frac{dT^*}{dr^*}\right)_a\left[r^*\frac{d}{dr^*}\left(\frac{u^*}{r^*}\right)\right]_a-\frac{9}{8}\left(\frac{dT^*}{dr^*}\frac{u^*}{r^*}\right)_a-\\
&\quad\frac{8}{15}T_a^*\left[\frac{1}{p^*}\frac{dp^*}{dr^*}r^*\frac{d}{dr^*}\left(\frac{u^*}{r^*}\right)\right]_a-\frac{5}{8}\left(\frac{1}{r^*}\frac{dT^*}{dr^*}\right)_a+\frac{5}{6}\gamma M_{wa}^2\left[u^*\frac{d}{dr^*}\left(\frac{u^*}{r^*}\right)\right]_a\\
X_b^*&\equiv\frac{a^2X_b}{RT_{wa}U}\\
&=-\frac{5}{6}T_b^*\left\{\frac{d}{dr^*}\left[r^*\frac{d}{dr^*}\left(\frac{u^*}{r^*}\right)\right]\right\}_b-\left(5.167-\frac{8}{15}\right)\left(\frac{dT^*}{dr^*}\right)_b\left[r^*\frac{d}{dr^*}\left(\frac{u^*}{r^*}\right)\right]_b-\frac{9}{8}\left(\frac{dT^*}{dr^*}\frac{u^*}{r^*}\right)_b-\\
&\quad\frac{8}{15}T_b^*\left[\frac{1}{p^*}\frac{dp^*}{dr^*}r^*\frac{d}{dr^*}\left(\frac{u^*}{r^*}\right)\right]_b\\
Z_a^*&\equiv\frac{a^2Z_a}{RT_{wa}^2}\\
&=e_1\gamma M_{wa}^2T_a^*\left[r^*\frac{d}{dr^*}\left(\frac{u^*}{r^*}\right)\right]_a^2+\frac{1}{2}\gamma M_{wa}^2T_a^*\left[u^*\frac{d}{dr^*}\left(\frac{u^*}{r^*}\right)\right]_a+e_3T_a^*\left(\frac{1}{r^*}\frac{dT^*}{dr^*}\right)_a+\\
&\quad\left(e_8-\frac{\beta}{14}\right)\left(\frac{dT^*}{dr^*}\right)_a^2-\frac{1}{14}T_a^*\left[\frac{d^2T^*}{(dr^*)^2}\right]_a+\gamma M_{wa}^2\left[\left(\frac{1}{2}+\frac{18}{7}u^*\right)\frac{1}{r^*}\frac{dT^*}{dr^*}\right]_a\\
Z_b^*&\equiv\frac{a^2Z_b}{RT_{wa}^2}\\
&=e_1\gamma M_{wa}^2T_b^*\left[r^*\frac{d}{dr^*}\left(\frac{u^*}{r^*}\right)\right]_b^2+\frac{1}{2}\gamma M_{wa}^2T_b^*\left[u^*\frac{d}{dr^*}\left(\frac{u^*}{r^*}\right)\right]_b+e_6T_b^*\left(\frac{1}{r^*}\frac{dT^*}{dr^*}\right)_b+\\
&\quad\left(e_8-\frac{\beta}{14}\right)\left(\frac{dT^*}{dr^*}\right)_b^2-\frac{1}{14}T_b^*\left[\frac{d^2T^*}{(dr^*)^2}\right]_b
\end{aligned}\right\} \quad \text{(C6)}$$

Upon using equations (89), (90), (91), and (C5), equations (C1) to (C4) become, respectively,

$$u_a^*=1+\frac{0.796a_1(1-k)}{km(T_a^*)^{1/2}}\left(\frac{d\omega^*}{d\zeta}\right)_0 Kn_e+\tilde{X}_a Kn_e^2 \tag{C7}$$

$$u_b^*=0-\frac{0.796a_1(1-k)(T_b^*)^{1/2}}{kmT_a^*p_b^*}\left(\frac{d\omega^*}{d\zeta}\right)_1 Kn_e+\tilde{X}_b Kn_e^2 \tag{C8}$$

$$T_a^*=1+\frac{0.796c_1(1-k)}{km(T_a^*)^{1/2}}\left(\frac{dT^*}{d\zeta}\right)_0 Kn_e+\tilde{Z}_a Kn_e^2 \tag{C9}$$

$$T_b^*=T_{wb}^*-\frac{0.796c_1(1-k)(T_b^*)^{1/2}}{mT_a^*p_b^*}\left(\frac{dT^*}{d\zeta}\right)_1 Kn_e+\tilde{Z}_b Kn_e^2 \tag{C10}$$

where

$$Kn_e=\eta Kn_{wa} \tag{C11}$$

and

$$\left.\begin{aligned}
\tilde{X}_a&\equiv\frac{0.6336(1-k)^2(\mu_a^*)^2X_a^*}{k^2\eta^2(T_a^*)^2}=\frac{0.6336(1-k)^2}{k^2m^2(T_a^*)^2}\Bigg[-\frac{5}{6}T^*\frac{d^2\omega^*}{d\zeta^2}-\left(4.634-\frac{5}{6}\beta\right)\frac{d\omega^*}{d\zeta}\frac{dT^*}{d\zeta}-\frac{8}{15}T^*\frac{d\omega^*}{d\zeta}\frac{d}{d\zeta}(\log_e p^*)-\\
&\quad\frac{m\mu_a^*}{\eta}\left(\frac{9}{8}\omega^*\frac{dT^*}{d\zeta}+\frac{5}{8}\frac{dT^*}{d\zeta}-\frac{5}{6}\gamma M_{wa}^2\omega^*\frac{d\omega^*}{d\zeta}\right)\Bigg]_{\zeta=0}\\
\tilde{X}_b&\equiv\frac{0.6336(1-k)^2(\mu_b^*)^2X_b^*}{k^2\eta^2(T_a^*)^2(p_b^*)^2}=\frac{0.6336(1-k)^2}{km^2(T_a^*)^2(p_b^*)^2}\Bigg[-\frac{5}{6}T^*\frac{d^2\omega^*}{d\zeta^2}-\left(4.634-\frac{5}{6}\beta\right)\frac{d\omega^*}{d\zeta}\frac{dT^*}{d\zeta}-\frac{8}{15}T^*\frac{d\omega^*}{d\zeta}\frac{d}{d\zeta}(\log_e p^*)-\\
&\quad\frac{9m\mu_b^*\omega^*}{8\eta}\frac{dT^*}{d\zeta}\Bigg]_{\zeta=1}\\
\tilde{Z}_a&\equiv\frac{0.6336(1-k)^2(\mu_a^*)^2Z_a^*}{k^2\eta^2(T_a^*)^2}=\frac{0.6336(1-k)^2}{k^2m^2(T_a^*)^2}\Bigg\{e_1\gamma M_{wa}^2T_a^*\left(\frac{d\omega^*}{d\zeta}\right)^2+e_8\left(\frac{dT^*}{d\zeta}\right)^2-\frac{1}{14}T^*\frac{d^2T^*}{d\zeta^2}+\\
&\quad\frac{m\mu_a^*}{\eta}\left[\frac{1}{2}\gamma M_{wa}^2T_a^*\omega^*\frac{d\omega^*}{d\zeta}+\left(\frac{1}{14}+e_6\right)T^*\frac{dT^*}{d\zeta}+\gamma M_{wa}^2\left(\frac{1}{2}+\frac{18}{7}\omega^*\right)\frac{dT^*}{d\zeta}\right]\Bigg\}_{\zeta=0}\\
\tilde{Z}_b&\equiv\frac{0.6336(1-k)^2(\mu_b^*)^2Z_b^*}{k^2\eta^2(T_a^*)^2(p_b^*)^2}=\frac{0.6336(1-k)^2}{k^2m^2(T_a^*)^2(p_b^*)^2}\Bigg\{e_1\gamma M_{wa}^2T^*\left(\frac{d\omega^*}{d\zeta}\right)^2+k^2e_8\left(\frac{dT^*}{d\zeta}\right)^2-\frac{1}{14}k^2T^*\frac{d^2T^*}{d\zeta^2}+\\
&\quad\frac{m\mu_b^*}{\eta}\left[\frac{1}{2}\gamma M_{wa}^2T^*\omega^*\frac{d\omega^*}{d\zeta}+\left(\frac{1}{14}+e_6\right)k^2T^*\frac{dT^*}{d\zeta}\right]\Bigg\}_{\zeta=1}
\end{aligned}\right\} \tag{C12}$$

because of equations (C6), (89), (90), (91), and (95).

From equations (102) and (98)

$$F'(\zeta)=f'(\zeta)=\sum_{n=1}^{\infty}\frac{(-2m)^{n+1}}{n!}k^{2\zeta}[\Delta(\zeta)]^n \tag{C13}$$

$$F''(\zeta)=\sum_{n=1}^{\infty}\frac{(-2m)^{n+1}}{n!}k^{2\zeta}[\Delta(\zeta)]^{n-1}[n\Delta'(\zeta)-2m\Delta(\zeta)] \tag{C14}$$

where the prime denotes the derivative with respect to ζ. Equation (93) gives

$$\Delta'(\zeta)=\frac{\mu^*}{\eta}-1 \tag{C15}$$

Therefore

$$\left.\begin{aligned}F'(0)&=0\\F'(1)&=0\end{aligned}\right\} \tag{C16}$$

$$\left.\begin{aligned}F''(0)&=4m^2\left(\frac{\mu_a^*}{\eta}-1\right)\\F''(1)&=4k^2m^2\left(\frac{\mu_b^*}{\eta}-1\right)\end{aligned}\right\} \tag{C17}$$

and from equations (104) and (100)

$$\begin{aligned}G'(\zeta)&=g'(\zeta)-g(1)+g(0)\\&=2mf(\zeta)-g(1)+g(0)\end{aligned} \tag{C18}$$

$$G''(\zeta)=2mf'(\zeta) \tag{C19}$$

whence

$$G''(0)=G''(1)=0 \tag{C20}$$

From equation (101), using equations (C16) and (C17),

$$\left.\begin{aligned}
\left(\frac{d\omega^*}{d\zeta}\right)_0&=\frac{\omega_a^*-\omega_b^*}{1-k^2-F(1)}(-2m)\\
\left(\frac{d\omega^*}{d\zeta}\right)_1&=\frac{\omega_a^*-\omega_b^*}{1-k^2-F(1)}(-2k^2m)\\
\left(\frac{d^2\omega^*}{d\zeta^2}\right)_0&=\frac{\omega_a^*-\omega_b^*}{1-k^2-F(1)}(4m^2\mu_a^*/\eta)\\
\left(\frac{d^2\omega^*}{d\zeta^2}\right)_0&=\frac{\omega_a^*-\omega_b^*}{1-k^2-F(1)}(4k^2m^2\mu_b^*/\eta)
\end{aligned}\right\} \tag{C21}$$

while from equation (103), using equation (C20),

$$\left.\begin{aligned}
\left(\frac{dT^*}{d\zeta}\right)_0 &= -(T_a^*-T_b^*)- \\
&\quad \frac{(\gamma-1)Pr_{wa}M_{wa}^2(\omega_a^*-\omega_b^*)^2}{[1-k^2-F(1)]^2}[1-k^2-2m-G'(0)] \\
\left(\frac{dT^*}{d\zeta}\right)_1 &= -(T_a^*-T_b^*)- \\
&\quad \frac{(\gamma-1)Pr_{wa}M_{wa}^2(\omega_a^*-\omega_b^*)^2}{[1-k^2-F(1)]^2}[1-k^2-2k^2m-G'(1)] \\
\left(\frac{d^2T^*}{d\zeta^2}\right)_0 &= -\frac{(\gamma-1)Pr_{wa}M_{wa}^2(\omega_a^*-\omega_b^*)^2}{[1-k^2-F(1)]^2}(4m^2) \\
\left(\frac{d^2T^*}{d\zeta^2}\right)_1 &= -\frac{(\gamma-1)Pr_{wa}M_{wa}^2(\omega_a^*-\omega_b^*)^2(4k^2m^2)}{[1-k^2-F(1)]^2}
\end{aligned}\right\} \tag{C22}$$

Putting equations (C21) and (C22) into equations (C7) to (C10) yields

$$\omega_a^*=1-1.592a_1\frac{\omega_a^*-\omega_b^*}{(T_a^*)^{1/2}}\frac{1-k}{k[1-k^2-F(1)]}Kn_e+\tilde{X}_aKn_e^2 \tag{C23}$$

$$\omega_b^*=0+1.592a_1\frac{(\omega_a^*-\omega_b^*)(T_b^*)^{1/2}}{T_a^*p_b^*}\frac{k^2(1-k)}{1-k^2-F(1)}Kn_e+k\tilde{X}_bKn_e^2 \tag{C24}$$

$$T_a^*=1-0.796c_1\frac{1-k}{km(T_a^*)^{1/2}}\left\{T_a^*-T_b^*+(\gamma-1)Pr_{wa}M_{wa}^2(\omega_a^*-\omega_b^*)^2\frac{1-k^2-2m-G'(0)}{[1-k^2-F(1)]^2}\right\}Kn_e+\tilde{Z}_aKn_e^2 \tag{C25}$$

$$T_b^*=T_{wb}^*+0.796c_1\frac{(1-k)(T_b^*)^{1/2}}{mT_a^*p_b^*}\left\{T_a^*-T_b^*+(\gamma-1)Pr_{wa}M_{wa}^2(\omega_a^*-\omega_b^*)^2\frac{1-k^2-2k^2m-G'(1)}{[1-k^2-F(1)]^2}\right\}\times Kn_e+\tilde{Z}_bKn_e^2 \tag{C26}$$

It is convenient to express ω_a^*, ω_b^*, T_a^*, T_b^*, u_a^*, u_b^*, p_b^*, and η in ascending powers of the effective Knudsen number Kn_e. Thus write

$$\left.\begin{aligned}
\omega_a^*&={}_0\omega_a^*(1+{}_1\omega_a^*Kn_e+{}_2\omega_a^*Kn_e^2)\\
\omega_b^*&={}_0\omega_b^*+{}_1\omega_b^*Kn_e+{}_2\omega_b^*Kn_e^2\\
T_a^*&={}_0T_a^*(1+{}_1T_a^*Kn_e+{}_2T_a^*Kn_e^2)\\
T_b^*&={}_0T_b^*(1+{}_1T_b^*Kn_e+{}_2T_b^*Kn_e^2)
\end{aligned}\right\} \tag{C27}$$

$$\left.\begin{aligned}
u_a^*&={}_0u_a^*(1+{}_1u_a^*Kn_e+{}_2u_a^*Kn_e^2)\\
u_b^*&={}_0u_b^*+{}_1u_b^*Kn_e+{}_2u_b^*Kn_e^2\\
p_b^*&={}_0p_b^*(1+{}_1p_b^*Kn_e+{}_2p_b^*Kn_e^2)\\
\eta&={}_0\eta(1+{}_1\eta^*Kn_e+{}_2\eta^*Kn_e^2)
\end{aligned}\right\} \tag{C28}$$

where the subscript 0, 1, or 2 written in front of a symbol denotes the order of the approximation to the boundary values.

Equations (C23) to (C26) give in successive steps the zero-order approximation:

$$\left.\begin{aligned}
{}_0\omega_a^*&=1\\
{}_0\omega_b^*&=0\\
{}_0T_a^*&=1\\
{}_0T_b^*&=T_{wb}^*
\end{aligned}\right\} \tag{C29}$$

the first-order correction terms:

$$\left.\begin{aligned}
{}_1\omega_a^*&=-\frac{1.592a_1(1-k)}{k[1-k^2-{}_0F(1)]}\\
{}_1\omega_b^*&=\frac{1.592a_1(1-k)k^2(T_{wb}^*)^{1/2}}{[1-k^2-{}_0F(1)]\,{}_0p_b^*}\\
{}_1T_a^*&=-\frac{0.796c_1(1-k)}{km}\left\{1-T_{wb}^*+(\gamma-1)Pr_{wa}M_{wa}^2\frac{1-k^2-2m-{}_0G'(0)}{[1-k^2-{}_0F(1)]^2}\right\}\\
{}_1T_b^*&=\frac{0.796c_1(1-k)}{m_0p_b^*(T_{wb}^*)^{1/2}}\left\{1-T_{wb}^*+(\gamma-1)Pr_{wa}M_{wa}^2\frac{1-k^2-2k^2m-{}_0G'(1)}{[1-k^2-{}_0F(1)]^2}\right\}
\end{aligned}\right\} \tag{C30}$$

where ${}_0F(1)$, ${}_0G'(0)$, and ${}_0G'(1)$ are obtained from equations (102) and (C18) using the zero-order approximation, equation (C29), to determine η and μ^*, and the second-order correction terms

$$\left.\begin{aligned}
{}_2\omega_a^*&=-\frac{1.592a_1(1-k)}{k[1-k^2-{}_0F(1)]}\left({}_1\omega_a^*-{}_1\omega_b^*-\frac{1}{2}\,{}_1T_a^*\right)+{}_0\tilde{X}_a\\
{}_2\omega_b^*&=\frac{1.592a_1(1-k)k^2(T_{wb}^*)^{1/2}}{[1-k^2-{}_0F(1)]{}_0p_b^*}\left({}_1\omega_a^*-{}_1\omega_b^*-{}_1T_a^*+\frac{1}{2}\,{}_1T_b^*-{}_1p_b^*\right)+k\,{}_0\tilde{X}_b\\
{}_2T_a^*&=-\frac{0.796c_1(1-k)}{km}\left\{\left(\frac{1}{2}\,{}_1T_a^*-\frac{1}{2}\,{}_1T_a^*T_{wb}^*-T_{wb}^*\,{}_1T_b^*\right)+(\gamma-1)PrM_{wa}^2\frac{1-k^2-2m-{}_0G'(0)}{[1-k^2-{}_0F(1)]^2}\left(2\,{}_1\omega_a^*-2\,{}_1\omega_b^*-\frac{1}{2}\,{}_1T_a^*\right)\right\}+{}_0\tilde{Z}_a\\
{}_2T_b^*&=\frac{0.796c_1(1-k)}{m_0p_b^*(T_{wb}^*)^{1/2}}\left\{T_{wb}^*\,{}_1T_a^*+\frac{1}{2}\left(1-3T_{wb}^*\right){}_1T_b^*-(1-T_{wb}^*){}_1p_b^*+(\gamma-1)PrM_{wa}^2\frac{1-k^2-2k^2m-{}_0G'(1)}{[1-k^2-{}_0F(1)]^2}\times\right.\\
&\quad\left.\left(2\,{}_1\omega_a^*-2\,{}_1\omega_b^*+\frac{1}{2}\,{}_1T_b^*-{}_1T_a^*-{}_1p_b^*\right)\right\}+\frac{{}_0\tilde{Z}_b}{T_{wb}^*}
\end{aligned}\right\} \tag{C31}$$

From equations (C12) and (C29), with $\mu^*=T^{*\beta}$,

$$\left.\begin{aligned}
{}_0\tilde{X}_a&=\frac{0.6336(1-k)^2}{k^2m^2}\Bigg\{-\frac{5}{6}\,{}_0\left(\frac{d^2\omega^*}{d\zeta^2}\right)_0-\left(4.634-\frac{5}{6}\beta\right){}_0\left(\frac{d\omega^*}{d\zeta}\right)_0{}_0\left(\frac{dT^*}{d\zeta}\right)_0-\\
&\qquad\frac{8}{15}\,{}_0\left(\frac{d\omega^*}{d\zeta}\right)_0{}_0\left(\frac{d\log_e p^*}{d\zeta}\right)_0+\frac{m}{{}_0\eta}\left[-\frac{7}{4}\,{}_0\left(\frac{dT^*}{d\zeta}\right)_0+\frac{5}{6}\gamma M_{wa}^2\,{}_0\left(\frac{d\omega^*}{d\zeta}\right)_0\right]\Bigg\}\\
{}_0\tilde{X}_b&=\frac{0.6336(1-k)^2}{km^2{}_0(p_b^*)^2}\left[-\frac{5}{6}T_{wb}^*\,{}_0\left(\frac{d^2\omega^*}{d\zeta^2}\right)_1-\left(4.634-\frac{5}{6}\beta\right){}_0\left(\frac{d\omega^*}{d\zeta}\right)_1{}_0\left(\frac{dT^*}{d\zeta}\right)_1-\frac{8}{15}T_{wb}^*\,{}_0\left(\frac{d\omega^*}{d\zeta}\right)_1{}_0\left(\frac{d\log_e p^*}{d\zeta}\right)_1\right]\\
{}_0\tilde{Z}_a&=\frac{0.6336(1-k)^2}{k^2m^2}\Bigg\{e_1\gamma M_{wa}^2\,{}_0\left(\frac{d\omega^*}{d\zeta}\right)_0^2+e_8\,{}_0\left(\frac{dT^*}{d\zeta}\right)_0^2-\frac{1}{14}\,{}_0\left(\frac{d^2T^*}{d\zeta^2}\right)_0+\frac{m}{{}_0\eta}\left[\frac{1}{2}\gamma M_{wa}^2\,{}_0\left(\frac{d\omega^*}{d\zeta}\right)_0+\left(e_6+\frac{1}{14}+\frac{43}{14}\gamma M_{wa}^2\right){}_0\left(\frac{dT^*}{d\zeta}\right)_0\right]\Bigg\}\\
{}_0\tilde{Z}_b&=\frac{0.6336(1-k)^2}{k^2m^2{}_0(p_b^*)^2}\left[e_1\gamma M_{wa}^2T_{wb}^*\,{}_0\left(\frac{d\omega^*}{d\zeta}\right)_1^2+k^2e_8\,{}_0\left(\frac{dT^*}{d\zeta}\right)_1^2-\frac{1}{14}k^2T_{wb}^*\,{}_0\left(\frac{d^2T^*}{d\zeta^2}\right)_1+\frac{k^2m\mu_{wb}^*}{{}_0\eta}\left(e_6+\frac{1}{14}\right)T_{wb}^*\,{}_0\left(\frac{dT^*}{d\zeta}\right)_1\right]
\end{aligned}\right\}\quad\text{(C32)}$$

Also equations (C21) and (C22) give, upon using equations (C29),

$$\left.\begin{aligned}
{}_0\left(\frac{d\omega^*}{d\zeta}\right)_0&=-\frac{2m}{1-k^2-{}_0F(1)}\\
{}_0\left(\frac{d\omega^*}{d\zeta}\right)_1&=-\frac{2k^2m}{1-k^2-{}_0F(1)}\\
{}_0\left(\frac{d^2\omega^*}{d\zeta^2}\right)_0&=\frac{4m^2}{[1-k^2-{}_0F(1)]{}_0\eta}\\
{}_0\left(\frac{d^2\omega^*}{d\zeta^2}\right)_1&=\frac{4k^2m^2\mu_{wb}^*}{[1-k^2-{}_0F(1)]{}_0\eta}
\end{aligned}\right\}\quad\text{(C33)}$$

$$\left.\begin{aligned}
{}_0\left(\frac{dT^*}{d\zeta}\right)_0&=-(1-T_{wb}^*)-\frac{(\gamma-1)Pr_{wa}M_{wa}^2}{[1-k^2-{}_0F(1)]^2}\left[1-k^2-2m-{}_0G'(0)\right]\\
{}_0\left(\frac{dT^*}{d\zeta}\right)_1&=-(1-T_{wb}^*)-\frac{(\gamma-1)Pr_{wa}M_{wa}^2}{[1-k^2-{}_0F(1)]^2}\left[1-k^2-2k^2m-{}_0G'(1)\right]\\
{}_0\left(\frac{d^2T^*}{d\zeta^2}\right)_0&=-\frac{(\gamma-1)Pr_{wa}M_{wa}^2}{[1-k^2-{}_0F(1)]^2}(4m^2)\\
{}_0\left(\frac{d^2T^*}{d\zeta^2}\right)_1&=-\frac{(\gamma-1)Pr_{wa}M_{wa}^2}{[1-k^2-{}_0F(1)]^2}(4k^2m^2)
\end{aligned}\right\}\quad\text{(C34)}$$

APPENDIX D

DETERMINATION OF APPROXIMATE EXPRESSIONS FOR DISTRIBUTIONS

From equations (90) and (91)

$$\eta=\int_0^1\mu^*d\zeta \qquad \text{(D1)}$$

$$\xi=\frac{1}{\eta}\int_0^\zeta\mu^*d\zeta \qquad \text{(D2)}$$

The kinetic theory of gases (ref. 4) gives

$$\left.\begin{aligned}\mu&\propto T^\beta\\ \text{that is,}\qquad \mu^*&=T^{*\beta}\end{aligned}\right\} \qquad \text{(D3)}$$

Experiment shows that for most gases the value of β lies between 0.5 and 1, and it varies slightly with temperature. Substituting equation (D3) into equations (D1), (D2), and (93) yields

$$\eta=\int_0^1 T^{*\beta}d\zeta \qquad \text{(D4)}$$

$$\xi=\frac{1}{\eta}\int_0^\zeta T^{*\beta}d\zeta \qquad \text{(D5)}$$

$$\Delta(\zeta)=\int_0^\zeta\left(\frac{T^{*\beta}}{\eta}-1\right)d\zeta \qquad \text{(D6)}$$

It is customary to assume β constant and for most gases its value is quite close to unity. While the calculations can be carried out for any constant value of β, it is much simpler to take $\beta=1$ as is done in the text from equation (124) on.

Substituting equation (103) into equation (D5) and taking $\beta=1$,

$$\xi=\frac{1}{\eta}\left\{T_a^*\zeta-\frac{1}{2}(T_a^*-T_b^*)\zeta^2+\frac{(\gamma-1)Pr_{wa}M_{wa}^2(\omega_a^*-\omega_b^*)^2}{[1-k^2-F(1)]^2}\times\left[\zeta-\frac{1}{2}(1-k^2)\zeta^2-\frac{1-k^{2\zeta}}{2m}+I(\zeta)\right]\right\} \tag{D7}$$

where $I(\zeta)=\int_0^\zeta G(\zeta)d\zeta$.

Substituting equation (103) into equation (D4), with $\beta=1$, gives

$$\eta=\frac{1}{2}(T_a^*+T_b^*)+(\gamma-1)Pr_{wa}M_{wa}^2\frac{(\omega_a^*-\omega_b^*)^2}{[1-k^2-F(1)]^2}\left[\frac{1}{2}(1+k^2)-\frac{1-k^2}{2m}+I(1)\right] \tag{D8}$$

Upon neglecting the rarefaction correction to $F(1)$ and $I(1)$ and using equations (C27), (C28), and (C29), this yields as successive approximations to η

$$_0\eta=\frac{1}{2}(1+T_{wb}^*)+\frac{(\gamma-1)Pr_{wa}M_{wa}^2}{[1-k^2-{}_0F(1)]^2}\left[\frac{1}{2}(1+k^2)-\frac{1-k^2}{2m}+{}_0I(1)\right] \tag{D9}$$

$$_0\eta\ {}_1\eta^*=\frac{1}{2}({}_1T_a^*+{}_1T_b^*)+2({}_1\omega_a^*-{}_1\omega_b^*)\frac{(\gamma-1)Pr\,M_{wa}^2}{[1-k^2-{}_0F(1)]^2}\left[\frac{1}{2}(1+k^2)-\frac{1-k^2}{2m}+{}_0I(1)\right] \tag{D10}$$

$$_0\eta\ {}_2\eta^*=\frac{1}{2}({}_2T_a^*+{}_2T_b^*)+[({}_1\omega_a^*-{}_1\omega_b^*)^2+2({}_2\omega_a^*-{}_2\omega_b^*)]\frac{(\gamma-1)Pr\,M_{wa}^2}{[1-k^2-{}_0F(1)]^2}\left[\frac{1}{2}(1+k^2)-\frac{1-k^2}{2m}+{}_0I(1)\right] \tag{D11}$$

Now $\Delta(0)=\Delta(1)=0$ and, for $0<\zeta<1$, $|\Delta(\zeta)|\ll 1$. As a first approximation assume (superscripts denote order of approximation to solutions of the differential equations regardless of the boundary conditions)

$$\Delta(\zeta)={}^1\Delta(\zeta)=0 \tag{D12}$$

It follows from equations (98), (100), (102), and (104) that

$${}^1f(\zeta)={}^1g(\zeta)={}^1F(\zeta)={}^1G(\zeta)=0 \tag{D13}$$

Substituting these into equations (101), (103), (D7), and (D8) gives

$${}^1\omega^*=\omega_a^*-\frac{\omega_a^*-\omega_b^*}{1-k^2}(1-k^{2\zeta}) \tag{D14}$$

$${}^1T^*=T_a^*-(T_a^*-T_b^*)\zeta+\frac{(\gamma-1)Pr_{wa}M_{wa}^2(\omega_a^*-\omega_b^*)^2}{(1-k^2)^2}[1-(1-k^2)\zeta-k^{2\zeta}] \tag{D15}$$

$${}^1\eta=\frac{1}{2}(T_a^*+T_b^*)+\frac{(\gamma-1)Pr\,M_{wa}^2(\omega_a^*-\omega_b^*)^2}{2(1-k^2)^2}\left(1+k^2-\frac{1-k^2}{m}\right) \tag{D16}$$

$${}^1\eta=\frac{1}{2}(T_a^*+T_b^*)+\frac{1}{12}(\gamma-1)Pr\,M_{wa}^2(\omega_a^*-\omega_b^*)^2\left(1+\frac{1}{2}\epsilon+\frac{19}{60}\epsilon^2+\frac{9}{40}\epsilon^3+\cdots\right) \tag{D17}$$

where

$$\epsilon=1-k^2 \tag{D18}$$

and

$${}^1\xi=\frac{1}{{}^1\eta}\left\{T_a^*\zeta-\frac{1}{2}(T_a^*-T_b^*)\zeta^2+\frac{(\gamma-1)PrM_{wa}^2(\omega_a^*-\omega_b^*)^2}{(1-k^2)^2}\left[\zeta-\frac{1}{2}(1-k^2)\zeta^2-\frac{1-k^{2\zeta}}{2m}\right]\right\} \tag{D19}$$

For the second approximation, equation (D6) gives

$${}^2\Delta(\zeta)=\int_0^\zeta\left(\frac{{}^1T^{*\beta}}{{}^1\eta}-1\right)d\zeta$$

which, upon using equations (D15) and (D16), yields for $\beta=1$

$${}^2\Delta(\zeta)=\frac{1}{2\,{}^1\eta}\left\{(T_a^*-T_b^*)\zeta(1-\zeta)+\frac{(\gamma-1)PrM_{wa}^2(\omega_a^*-\omega_b^*)^2}{(1-k^2)}\left[\zeta(1-\zeta)-\frac{1-\zeta+k^2\zeta-k^{2\zeta}}{(1-k^2)m}\right]\right\} \tag{D20}$$

where ${}^1\eta$ is given by equation (D16). It is noted that ${}^2\Delta(0)={}^2\Delta(1)=0$.

Neglecting the second and higher powers of $\Delta(\zeta)$, equation (98) yields

$${}^2f(\zeta)=4m^2\int k^{2\zeta}\,{}^2\Delta(\zeta)\,d\zeta$$

which, upon using equation (D20), gives

$${}^2f(\zeta)=\frac{k^{2\zeta}}{{}^1\eta}\left\{(T_a^*-T_b^*)\left[\frac{1-m}{2m}+(1-m)\zeta+m\zeta^2\right]+\frac{(\gamma-1)Pr_{wa}M_{wa}^2(\omega_a^*-\omega_b^*)^2}{(1-k^2)}\left[-m\zeta(1-\zeta)+\frac{1+k^2-k^{2\zeta}}{2(1-k^2)}\right]\right\}+\text{Constant} \tag{D21}$$

It follows from equation (102) that

$${}^2F(\zeta)=\frac{1}{{}^1\eta}\left\{(T_a^*-T_b^*)\left[\frac{1}{2}(1-m)(1-k^{2\zeta})+(1-m)\zeta k^{2\zeta}+m\zeta^2k^{2\zeta}\right]+\frac{(\gamma-1)PrM_{wa}^2(\omega_a^*-\omega_b^*)^2}{1-k^2}\left[-m\zeta(1-\zeta)k^{2\zeta}-\frac{(1-k^{2\zeta})(k^2-k^{2\zeta})}{2(1-k)^2}\right]\right\} \tag{D22}$$

and

$${}^2F(1)=\frac{1}{2\,{}^1\eta}(T_a^*-T_b^*)\left(1+k^2-\frac{1-k^2}{m}\right) \tag{D23}$$

$${}^2F(1)=\frac{\epsilon^2}{12\,{}^1\eta}(T_a^*-T_b^*)\left(1+\frac{1}{2}\epsilon+\frac{19}{60}\epsilon^2+\frac{9}{40}\epsilon^3+\ \ldots\right) \tag{D24}$$

where $\epsilon=1-k^2$ as before.

Substituting equation (D21) into equation (100) yields for Pr/c_p=Constant

$$
\begin{aligned}
{}^2g(\zeta)&=2m\int {}^2f(\zeta)d\zeta \\
&=\frac{k^{2\zeta}}{{}^1\eta}\left\{-(T_a^*-T_b^*)\left[\frac{3-m}{2m}+(2-m)\zeta+m\zeta^2\right]-\right. \\
&\quad\frac{(\gamma-1)PrM_{wa}^2(\omega_a^*-\omega_b^*)^2}{1-k^2}\left[\frac{1-k^2+2k^2m}{2(1-k^2)m}+\right. \\
&\quad\left.\left.(1-m)\zeta+m\zeta^2-\frac{k^{2\zeta}}{4(1-k^2)}\right]\right\}+{}_gC_0+{}_gC_1\zeta
\end{aligned}
\tag{D25}
$$

From equations (D25) and (104)

$$
\begin{aligned}
{}^2G(\zeta)=\frac{1}{{}^1\eta}\Bigg(&(T_a^*-T_b^*)\left\{\left[-\frac{3(1-k^2)}{2m}+1+k^2\right]\zeta+\right. \\
&\left.\left(\frac{3}{2m}-1\right)(1-k^{2\zeta})-(2-m)\zeta k^{2\zeta}-m\zeta^2k^{2\zeta}\right\}+ \\
&\frac{(\gamma-1)PrM_{wa}^2(\omega_a^*-\omega_b^*)^2}{1-k^2}\left[\left(-\frac{1-k^2}{2m}+\frac{1+k^2}{4}\right)\zeta+\right. \\
&\left(\frac{1}{2m}+\frac{k^2}{1-k^2}\right)(1-k^{2\zeta})-(1-m)\zeta k^{2\zeta}- \\
&\left.m\zeta^2k^{2\zeta}-\frac{1-k^{4\zeta}}{4(1-k^2)}\right]\Bigg)
\end{aligned}
\tag{D26}
$$

The integrals $I(\zeta)$ defined in equation (D7) for ξ and I (1) in equation (D8) for η can now be found from equation (D26). They are

$$
\begin{aligned}
I(\zeta)&=\int_0^\zeta {}^2G(\zeta)d\zeta \\
&=\frac{T_a^*-T_b^*}{{}^1\eta}\left\{\left(-1+\frac{3}{2m}\right)\zeta+\left[\frac{1}{2}(1+k^2)-\frac{3(1-k^2)}{4m}\right]\zeta^2+\right. \\
&\quad\left.\frac{3}{2m}\left(\frac{1}{2}-\frac{1}{m}\right)(1-k^{2\zeta})-\frac{1}{2}\left(1-\frac{3}{m}\right)\zeta k^{2\zeta}+\frac{1}{2}\zeta^2k^{2\zeta}\right\}+ \\
&\quad\frac{(\gamma-1)PrM_{wa}^2(\omega_a^*-\omega_b^*)^2}{{}^1\eta(1-k^2)}\left\{\left[-\frac{1-4k^2}{4(1-k^2)}+\frac{1}{2m}\right]\zeta+\right. \\
&\quad\left(\frac{1+k^2}{8}-\frac{1-k^2}{4m}\right)\zeta^2+\frac{1}{4m}\left(\frac{1-3k^2}{1-k^2}-\frac{3}{m}\right)(1-k^{2\zeta})- \\
&\quad\left.\left(\frac{1}{2}-\frac{1}{m}\right)\zeta k^{2\zeta}+\frac{1}{2}\zeta^2k^{2\zeta}+\frac{1-k^{4\zeta}}{16(1-k^2)m}\right\}
\end{aligned}
\tag{D27}
$$

$$
\begin{aligned}
I(1)&=\int_0^1 {}^2G(\zeta)d\zeta \\
&=\frac{1}{{}^1\eta}\left\{(T_a^*-T_b^*)\left[-\frac{1}{2}(1-k^2)+\frac{3(1+k^2)}{2m}-\frac{3(1-k^2)}{2m^2}\right]+\right. \\
&\quad(\gamma-1)PrM_{wa}^2(\omega_a^*-\omega_b^*)^2\left[\frac{-1+8k^2-k^4}{8(1-k^2)^2}+\right. \\
&\quad\left.\left.\frac{9(1+k^2)}{16(1-k^2)m}-\frac{3}{4m^2}\right]\right\}
\end{aligned}
\tag{D28}
$$

$$
\begin{aligned}
I(1)=-\frac{\epsilon^3}{120\,{}^1\eta}\Bigg[&(T_a^*-T_b^*)(1+\epsilon+\ .\ .\ .)+ \\
&(\gamma-1)PrM_{wa}^2(\omega_a^*-\omega_b^*)^2\left(0+\frac{1}{168}\epsilon+\ .\ .\ .\right)\Bigg]
\end{aligned}
\tag{D29}
$$

The derivatives $G'(0)$ and $G'(1)$ needed in the coefficients ${}_1T_a^*$, ${}_1T_b^*$, ${}_2T_a^*$, and ${}_2T_b^*$ in boundary conditions (C27) are also found from equation (D26) as follows:

$$
\begin{aligned}
{}^2G'(\zeta)=\frac{d}{d\zeta}{}^2G(\zeta)=\frac{1}{{}^1\eta}&\left\{(T_a^*-T_b^*)\left[1+k^2-\frac{3(1-k^2)}{2m}+(1-m)k^{2\zeta}+2m(1-m)\zeta k^{2\zeta}+2m^2\zeta^2k^{2\zeta}\right]+\right. \\
&\left.\frac{(\gamma-1)PrM_{wa}^2(\omega_a^*-\omega_b^*)^2}{1-k^2}\left[\frac{1+k^2}{4}-\frac{1-k^2}{2m}+\frac{1+k^2}{1-k^2}mk^{2\zeta}-2m^2\zeta(1-\zeta)k^{2\zeta}-\frac{m}{1-k^2}k^{4\zeta}\right]\right\}
\end{aligned}
\tag{D30}
$$

Thus

$$
{}^2G'(0)=\frac{1}{{}^1\eta}\left\{(T_a^*-T_b^*)\left[2+k^2-m-\frac{3(1-k^2)}{2m}\right]+(\gamma-1)PrM_{wa}^2(\omega_a^*-\omega_b^*)^2\left[\frac{1+k^2}{4(1-k^2)}+\frac{k^2m}{(1-k^2)^2}-\frac{1}{2m}\right]\right\}
\tag{D31}
$$

$$
{}^2G'(0)=\frac{\epsilon^3}{24\,{}^1\eta}\left[-(T_a^*-T_b^*)\left(1+\frac{11}{10}\epsilon+\frac{21}{20}\epsilon^2+\ .\ .\ .\right)+\frac{1}{30}(\gamma-1)PrM_{wa}^2(\omega_a^*-\omega_b^*)^2\left(1+\frac{3}{2}\epsilon+\ .\ .\ .\right)\right]
\tag{D32}
$$

and

$$
{}^2G'(1)=\frac{1}{{}^1\eta}\left\{(T_a^*-T_b^*)\left[1+2k^2+k^2m-\frac{3(1-k^2)}{2m}\right]+(\gamma-1)PrM_{wa}^2(\omega_a^*-\omega_b^*)^2\left[\frac{1+k^2}{4(1-k^2)}+\frac{k^2m}{(1-k^2)^2}-\frac{1}{2m}\right]\right\}
\tag{D33}
$$

$$ {}^2G'(1)=\frac{\epsilon^3}{24\,{}^1\eta}\left[(T_a{}^*-T_b{}^*)\left(1+\frac{9}{10}\,\epsilon+\frac{3}{4}\,\epsilon^2+\ .\ .\ .\right)+\frac{1}{30}\,(\gamma-1)\,Pr\,M_{wa}{}^2\,(\omega_a{}^*-\omega_b{}^*)^2\left(1+\frac{3}{2}\,\epsilon+\ .\ .\ .\right)\right] \tag{D34}$$

Using equation (D22) for ${}^2F(\zeta)$ and equation (D20) for ${}^2\Delta(\zeta)$, the integrals J_1 and J_2 in equations (128) become

$$ J_1(\zeta)={}^2F(\zeta)+4m\int_0^\zeta {}^2F(\zeta)\,d\zeta $$

$$ =-\frac{1}{{}^1\eta}\left\{(T_a{}^*-T_b{}^*)\left[2(1-m)\zeta+\frac{1}{2}\left(3-\frac{5}{m}\right)(1-k^{2\zeta})+(3-m)\,\zeta k^{2\zeta}+m\zeta^2k^{2\zeta}\right]+\frac{(\gamma-1)\,Pr\,M_{wa}{}^2\,(\omega_a{}^*-\omega_b{}^*)^2}{1-k^2}\left[\frac{2k^2m}{1-k^2}\,\zeta-\right.\right. $$

$$ \left.\left.\frac{1}{2}\left(\frac{2}{m}+\frac{3k^2-1}{1-k^2}\right)(1-k^{2\zeta})+m\zeta^2k^{2\zeta}+(2-m)\,\zeta k^{2\zeta}\right]\right\} \tag{D35}$$

$$ J_1(1)=-\frac{1}{{}^1\eta}\left\{(T_a{}^*-T_b{}^*)\left[\frac{7}{2}+\frac{3}{2}\,k^2-2m-\frac{5(1-k^2)}{2m}\right]+\right. $$

$$ \left.(\gamma-1)PrM_{wa}{}^2(\omega_a{}^*-\omega_b{}^*)^2\left[\frac{1+k^2}{2(1-k^2)}+\frac{2k^2m}{(1-k^2)^2}-\frac{1}{m}\right]\right\} \tag{D36}$$

$$ J_1(1)=\frac{\epsilon^2}{12\,{}^1\eta}\left[(T_a{}^*-T_b{}^*)\left(1+\frac{3}{2}\,\epsilon+\frac{17}{12}\,\epsilon^2+\frac{51}{40}\,\epsilon^3+\ .\ .\ .\right)-\right. $$

$$ \left.(\gamma-1)\,PrM_{wa}{}^2\,(\omega_a{}^*-\omega_b{}^*)^2\left(\frac{1}{30}\,\epsilon+\frac{1}{20}\,\epsilon^2+\ .\ .\ .\right)\right] \tag{D37}$$

$$ J_2(\zeta)=4k^4m^2\int_0^\zeta k^{-2\zeta}\,{}^2\Delta(\zeta)\,d\zeta $$

$$ =\frac{k^4}{{}^1\eta}\left\{(T_a{}^*-T_b{}^*)\left[-\frac{1}{2}\left(1+\frac{1}{m}\right)(k^{-2\zeta}-1)+(1+m)\,\zeta k^{-2\zeta}-\right.\right. $$

$$ \left.m\zeta^2k^{-2\zeta}\right]+\frac{(\gamma-1)\,PrM_{wa}{}^2\,(\omega_a{}^*-\omega_b{}^*)^2}{1-k^2}\left[\frac{2m}{1-k^2}\,\zeta-\right. $$

$$ \left.\left.\frac{1}{2}\left(\frac{3-k^2}{1-k^2}+\frac{2}{m}\right)(k^{-2\zeta}-1)+(2+m)\,\zeta k^{-2\zeta}-m\zeta^2k^{-2\zeta}\right]\right\} \tag{D38}$$

$$ J_2(1)=\frac{k^2}{{}^1\eta}\left\{(T_a{}^*-T_b{}^*)\,\frac{1}{2}\left(1+k^2-\frac{1-k^2}{m}\right)+(\gamma-1)\,PrM_{wa}{}^2(\omega_a{}^*-\omega_b{}^*)^2\left[\frac{1}{2}\left(\frac{1+k^2}{1-k^2}\right)+\frac{2k^2m}{(1-k^2)^2}-\frac{1}{m}\right]\right\} \tag{D39}$$

$$ J_2(1)=\frac{\epsilon^2}{12\,{}^1\eta}\left[(T_a{}^*-T_b{}^*)\left(1-\frac{1}{2}\,\epsilon-\frac{11}{60}\,\epsilon^2-\frac{11}{120}\,\epsilon^3+\ .\ .\ .\right)+(\gamma-1)\,PrM_{wa}{}^2(\omega_a{}^*-\omega_b{}^*)^2\,\frac{\epsilon}{30}\left(1+\frac{1}{2}\,\epsilon+\ .\ .\ .\right)\right] \tag{D40}$$

From equation (127), when use is made of equations (C29),

$$ {}_0^2J(\zeta)={}_0J_1(\zeta)+{}_0J_2(\zeta)-2k^{2-2\zeta}\,{}_0^2F(\zeta) $$

and so

$$ {}_0^2J(1)={}_0J_1(1)+{}_0J_2(1)-2\,{}_0^2F(1) $$

$$ =-\frac{1}{{}_0^1\eta}\left\{(1-T_{wb}{}^*)\left[\frac{9}{2}+2k^2-\frac{1}{2}\,k^4+2m-\frac{(1-k^2)(7-k^2)}{2m}\right]+(\gamma-1)\,PrM_{wa}{}^2\left[\frac{1}{2}\,(1+k^2)+\frac{2k^2m}{1-k^2}-\frac{1-k^2}{m}\right]\right\} \tag{D41}$$

$$ {}_0^2J(1)=\frac{\epsilon^4}{20\,{}_0^1\eta}\left[(1-T_{wb}{}^*)\left(1+\frac{11}{9}\,\epsilon+\ .\ .\ .\right)-(\gamma-1)\,PrM_{wa}{}^2\left(\frac{1}{18}+\frac{1}{12}\,\epsilon+\frac{143}{1512}\,\epsilon^2+\ .\ .\ .\right)\right] \tag{D42}$$

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